## Homework Assignment #5: Due Monday, October 23

**Part I: Apostol Chapter 6, pp. 143–145.** Exercises 13, 15, 17.

**Notes on Exercise 13:** (i) Although it's not specified in the exercise, you should assume that $G$ is abelian. (ii) It might actually be better to look at the sum

$$\sum_{r=1}^{m}\sum_{k=1}^{n} f_r(a^k)e^{-2\pi ik\ell/n},$$

where $\ell$ is an arbitrary integer between 0 and $n-1$.

**Exercise 13.** Let $f_1, f_2, \ldots, f_m$ be characters of a finite group $G$ of order $m$, and let $a$ be an element of $G$ of order $n$. Theorem 6.7 shows that each number $f_r(a)$ is an $n$th root of unity. Prove that every $n$th root of unity occurs equally often among the numbers $f_1(a), f_2(a), \ldots, f_m(a)$. [Hint: evaluate the sum

$$\sum_{r=1}^{m}\sum_{k=1}^{n} f_r(a^k)e^{-2\pi ik/n}$$

in two different ways to determine the number of times $e^{-2\pi ik/n}$ occurs.]

**SOLUTION:** On the one hand, we have

$$\sum_{r=1}^{m}\sum_{k=1}^{n} f_r(a^k)e^{-2\pi ik\ell/n} = \sum_{r=1}^{m}\sum_{k=1}^{n}\left(f_r(a)e^{-2\pi i\ell/n}\right)^k.$$

By standard formulas for a geometric sum, and the facts that $e^{-2\pi i\ell} = 1$ and $f_r(a^n) = 1$ (the latter because $a$ has order $n$), the sum on $k$, on the right, equals 0 unless $f_r(a)e^{-2\pi i\ell/n} = 1$, in which case the sum equals $n$. Consequently,

$$\sum_{r=1}^{m}\sum_{k=1}^{n} f_r(a^k)e^{-2\pi ik\ell/n} = \sum_{r=1}^{m}\begin{cases} n & \text{if } f_r(a) = e^{2\pi i\ell/n},\\ 0 & \text{if not}\end{cases}$$
$$= n\cdot\left|\{1 \le r \le m : f_r(a) = e^{2\pi i\ell/n}\}\right|.$$

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On the other hand,

$$\sum_{r=1}^{m}\sum_{k=1}^{n} f_r(a^k)e^{-2\pi ik\ell/n} = \sum_{k=1}^{n} e^{-2\pi ik\ell/n}\sum_{r=1}^{m} f_r(a^k).$$

The sum on $r$ equals $m$ if $a^k = e$ and 0 otherwise, by Theorem 6.13. But $a$ has order $n$, so the only integer $k$ between 1 and $n$ for which $a^k = e$ is the integer $k = n$. So

$$\sum_{r=1}^{m}\sum_{k=1}^{n} f_r(a^k)e^{-2\pi ik\ell/n} = e^{-2\pi in\ell/n}\cdot m = m.$$

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Comparing () with () gives

$$\left|\{1 \le r \le m : f_r(a) = e^{2\pi i\ell/n}\}\right| = \frac{m}{n}.$$

That is, each $n$th root of unity $e^{2\pi i\ell/n}$ occurs $m/n$ times among the numbers $f_1(a), f_2(a), \ldots, f_m(a)$.

**Exercise 15.** Let $\chi$ be any nonprincipal Dirichlet character mod $k$. Prove that for all integers $a < b$ we have

$$\left|\sum_{n=a}^{b} \chi(n)\right| \leq \frac{1}{2}\varphi(k).$$

**SOLUTION:** Break the set of integers $S = \{a, a+1, \ldots, b-1, b\}$ up into intervals of length $k$, plus whatever is left over at the tail end on the right. That is, write

$$\sum_{n=a}^{b} \chi(n) = \sum_{r=0}^{M-1} \sum_{n=a+rk}^{a+(r+1)k-1} \chi(n) + \sum_{n=a+Mk}^{b} \chi(n)$$

for some nonnegative integer $M$. (If $M = 0$ then the above sum on $r$ is empty.) Now every set of $k$ consecutive integers contains exactly one representative of each equivalence class mod $k$. That is, for any integer $r$,

$$\left\{\overline{a+rk}, \overline{a+rk+1}, \ldots, \overline{a+(r+1)k-1}\right\} = \mathbb{Z}/k\mathbb{Z}.$$

So every such set of integers must contain exactly one representative of each element of $(\mathbb{Z}/kZ)^*$. But then the sum of $\chi(n)$, over such a set of integers $n$, must be zero, by Theorem. So our above formula for the sum from $a$ to $b$ gives

$$\sum_{n=a}^{b} \chi(n) = \sum_{n=a+Mk}^{b} \chi(n).$$

The sum on the right contains at most $\varphi(k)$ nonzero terms. If it contains $\leq \varphi(k)/2$ nonzero terms, then we're done, since each such term has absolute value 1, so the sum of such terms, in absolute value, is $\leq \varphi(k)/2$. If the sum from $a + Mk$ to $b$ has more than $\varphi(k)/2$ nonzero terms, then the sum from $b+1$ to $a+(M+1)k-1$ has at most that many nonzero terms. But in this case, we note that

$$\sum_{n=a+Mk}^{b} \chi(n) = \sum_{n=a+Mk}^{a+(M+1)k-1} \chi(n) - \sum_{n=b+1}^{a+(M+1)k-1} \chi(n) = 0 - \sum_{n=b+1}^{a+(M+1)k-1} \chi(n),$$

and since the remaining sum on the right is bounded in absolute value by $\varphi(k)/2$ (as it has at most $\varphi(k)/2$ nonzero terms, each of absolute value one), so is the sum on the left, and were done.

**Exercise 17.** An arithmetic function $f$ is called *periodic* mod $k$ if $k > 0$ and $f(m) = f(n)$ whenever $m \equiv n \pmod{k}$. The integer $k$ is called the *period* of $f$.

**(a)** If $f$ is periodic mod $k$, prove that $f$ has a smallest positive period $k_0$, and that $k_0 | k$.

**(b)** Let $f$ be completely multiplicative and periodic, let $k$ be the smallest positive period of $f$. Prove that $f(n) = 0$ if $(n, k) > 1$. This shows that $f$ is a Dirichlet character mod $k$.

**SOLUTION:**

**(a)** Under the stated conditions, the set of positive periods of $f$ is nonempty because it contains $k$. So by the well-ordering principle, this set has a smallest positive element $k_0$. To show that

$k_0|k$, suppose not. Then we can write $k = k_0 q + r$ where $0 < r < k_0$. Now let $m$ and $n$ be any two integers that are congruent mod $r$. Then there is an integer $\ell$ such that

$$m = n + \ell r = n + \ell(k - k_0 q).$$

But then

$$f(m) = f(n + \ell(k - k_0 q)) = f(n + \ell k - \ell k_0 q) = f(n + \ell k) = f(n),$$

so $f$ has period $r$. This contradicts the fact that $k_0$ is the smallest positive period of $f$. So we must have $k_0|k$.

**(b)** Let $f$ and $k$ be as stated. Let $n$ be an integer with $f(n) \neq 0$ and $(n, k) = d > 1$. Write $n = da$ and $k = db$ for integers $a$ and $b$, with $b < k$ (since $d > 1$). We have $f(n) = f(da) = f(d)f(a)$, and since $f(n) \neq 0$, we have $f(d) \neq 0$. Then for any integer $m$, we have

$$f(d)f(m) = f(dm) = f(dm + k) = f(dm + db) = f(d)f(m + b)$$

or, dividing by $f(d)$, $f(m) = f(m + b)$. This contradicts the minimality of $k$. So we must have $f(n) = 0$ for $(n, k) > 1$.

**Part II.** **(A)** Evaluate (as a real number) the series

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n},$$

where $\chi$ is the unique nontrivial Dirichlet character mod 3. Hint: after writing the series out explicitly, consider the integral $\int_0^1 t^{3n}(1-t)\,dt$. Note that tables of values of Dirichlet characters are given on p. 139 of Apostol.

**SOLUTION:** We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\chi(n)}{n} &= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots \ldots \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{3n+1} - \frac{1}{3n+2}\right) = \sum_{n=0}^{\infty}\left(\int_0^1 t^{3n}\,dt - \int_0^1 t^{3n+1}\,dt\right) = \sum_{n=0}^{\infty}\int_0^1 t^{3n}(1-t)\,dt \\
&= \int_0^1 \sum_{n=0}^{\infty} t^{3n}(1-t)\,dt = \int_0^1 \frac{1-t}{1-t^3}\,dt = \int_0^1 \frac{dt}{t^2+t+1} = \int_0^1 \frac{dt}{(t+\frac{1}{2})^2 + \frac{3}{4}} \\
&= \frac{2}{\sqrt{3}} \arctan\left(\frac{1+2t}{\sqrt{3}}\right)\Bigg|_0^1 = \frac{2}{\sqrt{3}}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3\sqrt{3}}.
\end{aligned}$$

**(B)** Repeat **(A)** above for the unique *real-valued*, nonprincipal Dirichlet character mod 5. The series should end up as an integral of a rational function. Do the best you can with this integral: leave as is, evaluate numerically or, if possible, evaluate explicitly.

**SOLUTION:** We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\chi(n)}{n} &= \frac{1}{1}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots\ ... \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{5n+1}-\frac{1}{5n+2}-\frac{1}{5n+3}+\frac{1}{5n+4}\right) = \sum_{n=0}^{\infty}\int_0^1 \left(t^{5n}-t^{5n+1}-t^{5n+2}+t^{5n+3}\right)dt \\ &= \int_0^1 \sum_{n=0}^{\infty} t^{5n}\big(1-t-t^2(1-t)\big)\,dt = \int_0^1 \frac{\big(1-t-t^2(1-t)\big)\,dt}{1-t^5} = \int_0^1 \frac{(1-t^2)\,dt}{1+t+t^2+t^3+t^4}.\end{aligned}$$

I don't know how to do the latter integral, but Mathematica tells me it's about 0.430409.