Homework Assignment #4: Solutions to Selected Exercises

Part I. Apostol Chapter 3:

Exercise 2. If $x \geq 2$, prove that

$$\sum_{n\leq x} \frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2C\log x + O(1),$$

where C is Euler's constant.

SOLUTION: Note that

$$\frac{d(n)}{n} = \frac{1}{n}\sum_{d|n} 1 = \sum_{d|n} \frac{1}{d}\cdot\frac{d}{n}\cdot = r * r(n),$$

where $r(n) = 1/n$. By Theorem 3.10, then,

$$\sum_{n\leq x} \frac{d(n)}{n} = \sum_{n\leq x} r * r(n) = \sum_{n\leq x} r(n) R\left(\frac{x}{n}\right) = \sum_{n\leq x} \frac{1}{n} R\left(\frac{x}{n}\right),$$

where

$$R(x) = \sum_{n\leq x} r(n) = \sum_{n\leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right),$$

the last step by Theorem 3.2(a). So

$$\begin{aligned}\sum_{n\leq x} \frac{d(n)}{n} &= \sum_{n\leq x} \frac{1}{n}\left(\log\left(\frac{x}{n}\right) + C + O\left(\frac{n}{x}\right)\right)\\ &= \log x \sum_{n\leq x}\frac{1}{n} - \sum_{n\leq x}\frac{\log n}{n} + C\sum_{n\leq x}\frac{1}{n} + O\left(\frac{1}{x}\sum_{n\leq x} 1\right)\\ &= \log x \sum_{n\leq x}\frac{1}{n} - \sum_{n\leq x}\frac{\log n}{n} + C\sum_{n\leq x}\frac{1}{n} + O(1).\end{aligned}$$

By Theorem 3.2(a) again, and by Exercise 1(a) from Chapter 3, we then have, for some constant A,

$$\begin{aligned}\sum_{n\leq x} \frac{d(n)}{n} &= \log x\left(\log x + C + O\left(\frac{1}{x}\right)\right)\\ &\quad - \left(\frac{1}{2}\log^2 x + A + O\left(\frac{\log x}{x}\right)\right) + C\left(\log x + C + O\left(\frac{1}{x}\right)\right) + O(1)\\ &= \frac{1}{2}\log^2 x + 2C\log x + O(1).\end{aligned}$$

Part II.

(A) Prove that, if $s > 1$, then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Hint: prove that

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1,$$

by expanding the zeta function as a sum over m.

(B) Apostol Chapter 3 (pp. 70–73): Exercises 4(a), 5(a). (You may want to use part (A) directly above.)

(C) (This is easy.) Use parts (A,B) above to verify the first "=" given at the bottom of p. 70.

SOLUTION:

(A)

$$\begin{aligned}
\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} &= \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(n)}{(mn)^s} \\
&= \sum_{k=1}^{\infty} \sum_{mn=k} \frac{\mu(n)}{(mn)^s} = \sum_{k=1}^{\infty} \sum_{n|k} \frac{\mu(n)}{k^s} = \sum_{k=1}^{\infty} \Bigg(\sum_{n|k} \mu(n) \Bigg) \frac{1}{k^s} \\
&= \sum_{k=1}^{\infty} \left[\frac{1}{k} \right] \frac{1}{k^s} = \frac{1}{1^s} = 1,
\end{aligned}$$

the next-to-last equality by Theorem 2.1.

(B): Exercise 4(a). If $x \geq 2$ prove that

$$\sum_{n \leq x} \mu(n) \left[\frac{x}{n} \right]^2 = \frac{x^2}{\zeta(2)} + O(x \log x).$$

SOLUTION:

$$\begin{aligned}
\sum_{n \leq x} \mu(n) \left[\frac{x}{n} \right]^2 &= \sum_{n \leq x} \mu(n) \left(\frac{x}{n} + O(1) \right)^2 = x^2 \sum_{n \leq x} \frac{\mu(n)}{n^2} + O\Bigg(\sum_{n \leq x} \frac{x}{n} \Bigg) + O\Bigg(\sum_{n \leq x} 1 \Bigg) \\
&= x^2 \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - x^2 \sum_{n > x} \frac{\mu(n)}{n^2} + O\Bigg(\sum_{n \leq x} \frac{x}{n} \Bigg) + O\Bigg(\sum_{n \leq x} 1 \Bigg) \\
&= \frac{x^2}{\zeta(2)} + O\Bigg(x^2 \sum_{n > x} \frac{1}{n^2} \Bigg) + O\Bigg(x \sum_{n \leq x} \frac{1}{n} \Bigg) + O\Bigg(\sum_{n \leq x} 1 \Bigg) \\
&= \frac{x^2}{\zeta(2)} + O(x) + O(x \log x) + O(x) = \frac{x^2}{\zeta(2)} + O(x \log x),
\end{aligned}$$

the third-to-last equality by part (A) above, and the next-to-last by Theorem 3.2(a) and (b).

(B): Exercise 5(a). If $x \geq 1$ prove that

$$\sum_{n \leq x} \varphi(n) = \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right]^2 + \frac{1}{2}.$$

SOLUTION:

$$\sum_{n \leq x} \varphi(n) = \sum_{n \leq x} \mu * N(n)$$

by Theorem 2.3, where $N(n) = n$. So by Theorem 3.10,

$$\begin{aligned}
\sum_{n \leq x} \varphi(n) &= \sum_{n \leq x} \mu(n) \sum_{m \leq x/n} N(m) = \sum_{n \leq x} \mu(n) \sum_{m \leq x/n} m \\
&= \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] \left(\left[\frac{x}{n}\right] + 1\right) = \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right]^2 + \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] \\
&= \frac{1}{2} \sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right]^2 + \frac{1}{2},
\end{aligned}$$

the last step by Theorem 3.12(a).

(C) SOLUTION: This is just the two formulas from part (B) combined.

Part III. Apostol Chapter 4 (pp. 101–105):

Exercise 20.

Show that, if f is an a.f. with

$$\sum_{p \leq x} f(p) \log p = (ax + b) \log x + cx + O(1)$$

for positive constants a, b, c (the sum is over *primes* p), then there is a constant A (depending on f) such that

$$\sum_{p \leq x} f(p) = ax + (a + c)\left(\frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}\right) + b \log(\log x) + A + O\left(\frac{1}{\log x}\right).$$

SOLUTION: It might be better to rename the a.f. f in question "g," and reserve "f" for the function that appears in ASF. So we are given

$$\sum_{n \leq x} \Lambda_1(n)\, g(n) \log n = (ax + b) \log x + cx + h(x)$$

where $h(x)$ $(=\sum_{p\leq x} f(p)\log p - (ax+b)\log x - cx)$ is $O(1)$, and

$$\Lambda_1(n) = \begin{cases} 1 & \text{if } n \text{ is prime;} \\ 0 & \text{if not.} \end{cases}$$

Now we apply ASF, with $y = 2$, $f(t) = 1/\log t$, and

$$A(x) = \sum_{p\leq x} g(p)\log p = \sum_{n\leq x} \Lambda_1(n)\, g(n)\log n = (ax+b)\log x + cx + h(x),$$

to get

$$\begin{aligned}
\sum_{n\leq x} \Lambda_1(n)\, g(n) &= \Lambda_1(2)\, g(2) + \sum_{2<n\leq x} \Lambda_1(n)\, g(n) = \Lambda_1(2)\, g(2) + \frac{A(x)}{\log x} - \frac{A(2)}{\log 2} - \int_2^x A(t)\, f'(t)\, dt \\
&= \Lambda_1(2)\, g(2) + \frac{(ax+b)\log x + cx + h(x)}{\log x} - \Lambda_1(2)\, g(2) + \int_2^x \frac{(at+b)\log t + ct + h(t)}{t\log^2 t}\, dt \\
&= (ax+b) + \frac{cx}{\log x} + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{(at+b)\log t + ct}{t\log^2 t}\, dt + \int_2^\infty \frac{h(t)}{t\log^2 t}\, dt - \int_x^\infty \frac{h(t)}{t\log^2 t}\, dt \\
&= ax + \frac{cx}{\log x} + B + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{(at+b)\log t + ct}{t\log^2 t}\, dt + O\left(\int_x^\infty \frac{dt}{t\log^2 t}\right) \\
&= ax + \frac{cx}{\log x} + B + O\left(\frac{1}{\log x}\right) + a\int_2^x \frac{dt}{\log t} + b\int_2^x \frac{dt}{t\log t} + c\int_2^x \frac{dt}{\log^2 t} + O\left(-\log^{-1} t\,\Big|_x^\infty\right) \\
&= ax + \frac{cx}{\log x} + B + a\left(\frac{t}{\log t}\Big|_2^x + \int_2^x \frac{dt}{t\log^2 t}\right) + b\int_{\log 2}^{\log x} \frac{du}{u} + c\int_2^x \frac{dt}{\log^2 t}\, dt + O\left(\frac{1}{\log x}\right) \\
&= ax + (a+c)\left(\frac{x}{\log x} + \int_2^x \frac{dt}{t\log^2 t}\right) + b\log\log x + A + O\left(\frac{1}{\log x}\right)
\end{aligned}$$

where

$$A = b(1 - \log\log 2) - \frac{2a}{\log 2} + \int_2^\infty \frac{h(t)}{t\log^2 t}.$$

Part IV.

(1) Prove that, if

$$\sum_{n\leq x} a(n) = Ax^u \log^v x + O(x^u \log^{v-1} x)$$

for $a(n)$ an arithmetic function, A a constant, and $u, v > 1$, then

$$\sum_{n\leq x} \frac{a(n)}{n} = A\left(1 + \frac{1}{u-1}\right) x^{u-1}\log^v x + O(x^{u-1}\log^{v-1} x).$$

(2) Deduce the appropriate conclusion regarding $\sum a(n)/n$ if, in the hypotheses of problem **(1)** above, we replace "$u > 1$" with "$u = 1$."

SOLUTION:

(1) Applying ASF (Theorem 4.2) with $f(x) = 1/x$ and $y = 1/2$, say, and noting that $A(t) = \sum_{n\le t} a(n)/n = 0$ for $t < 1$, we have

$$\begin{aligned}
&\sum_{n\le x}\frac{a(n)}{n}\\
&= \frac{1}{x}\Big(Ax^u\log^v x + O(x^u\log^{v-1}x)\Big) - 0\cdot f\left(\frac{1}{2}\right) - \int_1^x \frac{-1}{t^2}\Big(At^u\log^v t + O(t^u\log^{v-1}t)\Big)\,dt\\
&= Ax^{u-1}\log^v x + O(x^{u-1}\log^{v-1}x) + A\int_1^x t^{u-2}\log^v t\,dt + O\bigg(\int_1^x t^{u-2}\log^{v-1}t\,dt\bigg).
\end{aligned}$$

Integrating by parts, in the first integral, with $p = \log^v t$ and $dq = t^{u-2}\,dt$, gives

$$\begin{aligned}
\sum_{n\le x}\frac{a(n)}{n} &= Ax^{u-1}\log^v x + O(x^{u-1}\log^{v-1}x) + A\left[\frac{t^{u-1}}{u-1}\log^v t\Big|_1^x - \frac{v}{u-1}\int_1^x t^{u-2}\log^{v-1}t\,dt\right]\\
&+ O\bigg(\int_1^x t^{u-2}\log^{v-1}t\,dt\bigg).
\end{aligned}$$

Now note that, because $\log^{v-1}t$ is increasing and nonnegative on $[1, x]$, we have

$$\int_1^x t^{u-2}\log^{v-1}t\,dt \le \log^{v-1}x\int_1^x t^{u-2}\,dt = O(x^{u-1}\log^{v-1}x).$$

The desired result follows.

(2) I leave this result to you.