Homework Assignment #3: Solutions to Selected Exercises

Part I. Apostol Chapter 2 (pp. 46–51): Exercises 9, 26.

Exercise 9. If x is real, $x \geq 1$, let $\varphi(x, n)$ denote the number of positive integers $\leq x$ that are relatively prime to n. [Note that $\varphi(n, n) = \varphi(n)$.] Prove that

$$\varphi(x, n) = \sum_{d|n} \mu(d) \left[\frac{x}{d}\right] \quad \text{and} \quad \sum_{d|n} \varphi\left(\frac{x}{d}, \frac{n}{d}\right) = [x].$$

SOLUTION: We prove the second identity first. We have

$$\sum_{d|n} \varphi\left(\frac{x}{d}, \frac{n}{d}\right) = \sum_{d|n} \sum_{\substack{k \leq x/d \\ (k, n/d)=1}} 1.$$ ()

Now consider the set

$$A(n, x) = \left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{[x]}{n}\right\} = \left\{\frac{m}{n} \middle| 1 \leq m \leq x\right\}.$$

Consider a fraction m/n in this set: we can put this fraction into reduced form by dividing both m and n by $d = (m, n)$. We then get a fraction $k/(n/d)$, where $d|n$ and $(k, n/d) = 1$. Also, since $m \leq x$, we have $k = m/d \leq x/d$. This correspondence $m/n \leftrightarrow k/(n/d)$ is easily seen to be injective and surjective, so we find that

$$[x] = \left|A(n, x)\right| = \sum_{d|n} \sum_{\substack{k \leq x/d \\ (k, n/d)=1}} 1,$$

so () gives us the second identity in question.

Now let's replace x by xn in that identity, to get

$$\sum_{d|n} \varphi\left(x \cdot \frac{n}{d}, \frac{n}{d}\right) = [xn].$$

Replacing d by n/d in the sum gives

$$\sum_{d|n} \varphi(xd, d) = [xn].$$

By Möbius inversion, then,

$$\sum_{d|n} \mu(d) \left[\frac{xn}{d}\right] = \varphi(xn, n).$$

Replacing x by x/n then gives us the first of the desired identities, and we're done.

Exercise 26. Assume f is multiplicative. Show f is completely multiplicative if, and only if, $f^{-1}(p^a) = 0$ for all primes p and all integers $a \geq 2$.

SOLUTION: If f is completely multiplicative then, by Theorem 2.17, we have $f^{-1}(n) = \mu(n)f(n)$ for all positive integers n, so for p prime and $a \geq 2$,

$$f^{-1}(p^a) = \mu(p^a)f(p^a) = 0 \cdot f(p^a) = 0,$$

and the desired result holds.

Conversely, suppose f is multiplicative and $f^{-1}(p^a) = 0$ for all primes p and all integers $a \geq 2$. Then for such p and a, we have, by the formula for Dirichlet inverses,

$$0 = f^{-1}(p^a) = \frac{-1}{f(1)} \sum_{b=0}^{a-1} f^{-1}(p^b)f(p^{a-b}) = -f^{-1}(1)f(p^a) - f^{-1}(p)f(p^{a-1}) = -f(p^a) - f^{-1}(p)f(p^{a-1}),$$

so $f(p^a) = -f^{-1}(p)f(p^{a-1})$. Moreover, $f^{-1}(p) = -f(p)$ by this same formula. So $f(p^a) = f(p)f(p^{a-1})$. An inductive argument on a then shows that $f(p^a) = f(p)^a$. So f is completely multiplicative, by Theorem 2.13(b), and we're done.

Part II. Apostol Chapter 3 (pp. 70–73): Exercise 1. Use Euler's summation formula to deduce the following for $x \geq 2$:

(a)
$$\sum_{2 \leq n \leq x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + A + O\left(\frac{\log x}{x}\right),$$

where A is constant;

(b)
$$\sum_{2 \leq n \leq x} \frac{1}{n \log n} = \log(\log x) + B + O\left(\frac{1}{x \log x}\right),$$

where B is constant.

SOLUTION: We'll do part (b); part (a) is similar.

We have, by ESF′ with $f(t) = 1/(t \log t)$ and $y = 2$,

$$\begin{aligned}
&\sum_{2 \leq n \leq x} \frac{1}{n \log n} \\
&= \frac{([x] - x)}{x \log x} + \frac{1}{2 \log 2} + \int_2^x \frac{dt}{t \log t} + \int_2^x \frac{(t - [t])(1 + \log t)}{t^2 \log^2 t}\, dt \\
&= O\left(\frac{1}{x \log x}\right) + \frac{1}{2 \log 2} + \log\log x - \log\log 2 + \int_2^x \frac{(t - [t])(1 + \log t)}{t^2 \log^2 t}\, dt \\
&= O\left(\frac{1}{x \log x}\right) + \frac{1}{2 \log 2} + \log\log x - \log\log 2 + \int_2^\infty \frac{(t - [t])(1 + \log t)}{t^2 \log^2 t}\, dt \\
&- \int_x^\infty \frac{(t - [t])(1 + \log t)}{t^2 \log^2 t}\, dt \\
&= O\left(\frac{1}{x \log x}\right) + \frac{1}{2 \log 2} + \log\log x - \log\log 2 + \int_2^\infty \frac{(t - [t])(1 + \log t)}{t^2 \log^2 t}\, dt + O\left(\int_x^\infty \frac{1 + \log t}{t^2 \log^2 t}\, dt\right)
\end{aligned}$$

since $\left|t-[t]\right|<1$. Now by the substitution $u=1/(t\log t)$,

$$\int_x^\infty \frac{1+\log t}{t^2\log^2 t}\,dt=\frac{1}{x\log x},$$

so

$$\begin{aligned}\sum_{2\le n\le x}\frac{1}{n\log n}&=O\left(\frac{1}{x\log x}\right)+\frac{1}{2\log 2}+\log\log x-\log\log 2+\int_2^\infty \frac{(t-[t])(1+\log t)}{t^2\log^2 t}\,dt\\&=\log\log x+B+O\left(\frac{1}{x\log x}\right),\end{aligned}$$

where

$$B=\frac{1}{2\log 2}-\log\log 2+\int_2^\infty \frac{(t-[t])(1+\log t)}{t^2\log^2 t}\,dt;$$

this last integral is easily seen to converge absolutely, since the integrand is $O(1/t^2)$.