Homework Assignment #2: Solutions to Selected Exercises

Part I. Apostol Chapter 2 (pp. 46–51): Exercises 3, 4, 5, 14, 29.

Exercise 3. Prove that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}. \qquad (🐟)$$

SOLUTION: It's readily seen that a quotient of multiplicative functions is multiplicative. So the left side of (🐟) is multiplicative. Further, the right side of (🐟) equals $(\mu^2/\varphi) * u$, and μ, φ, and u are all multiplicative, so $(\mu^2/\varphi) * u$ is too, by Theorem 2.14. Also, both sides of (🐟) equal 1 if $n = 1$. So it's enough to prove (🐟) for $n = p^\alpha$, where p is a prime and $\alpha \in \mathbb{Z}_+$.

Now

$$\frac{p^\alpha}{\varphi(p^\alpha)} = \frac{p^\alpha}{p^\alpha(1-1/p)} = \frac{1}{1-1/p}.$$

On the other hand, since $\mu(p^k) = 0$ for $k \geq 2$,

$$\sum_{d|p^\alpha} \frac{\mu^2(d)}{\varphi(d)} = \sum_{k=0}^{\alpha} \frac{\mu^2(p^k)}{\varphi(p^k)} = \frac{\mu^2(1)}{\varphi(1)} + \frac{\mu^2(p)}{\varphi(p)} = 1 + \frac{1}{p-1} = \frac{p}{p-1} = \frac{1}{1-1/p},$$

and we're done.

Exercise 4. Prove that $\varphi(n) > n/6$ for all n with at most 8 distinct prime factors.

SOLUTION: We have

$$\frac{\varphi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right)$$

by Theorem 2.4. Now the fewer factors there are on the right side, the larger that right side is, since each factor is < 1. Also, the larger p is, the larger $1 - 1/p$ is. So the right side is at least as large as the product you get when p ranges over the first 8 primes. So

$$\frac{\varphi(n)}{n} \geq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{19}\right) \approx 0.171024 > \frac{1}{6},$$

and we're done.

Exercise 5. Define $\nu(1) = 0$ and for $n > 1$ let $\nu(n)$ be the number of distinct prime factors of n. Let $f = \mu * \nu$ and prove that $f(n)$ is either 0 or 1.

SOLUTION: The statement $f = \mu * \nu$ is equivalent, by Möbius inversion, to the statement $\nu = f * u$. So it suffices to show that, for some function f with $f(n)$ always equal to 0 or 1, we have

$$\nu(n) = f * u(n) = \sum_{d|n} f(n).$$

The function f defined by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{in not} \end{cases}$$

works, since $f(n)$ equals 0 or 1, and

$$\sum_{d|n} f(d) = \sum_{p|n} 1 = \nu(n).$$

Exercise 14. Let $f(x)$ be defined for all x in $0 \le x \le 1$ and let

$$F(n) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right); \qquad F^*(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\left(\frac{k}{n}\right).$$

(a) Prove that $F^* = \mu * F$.

(b) Prove that

$$\mu(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} \mathrm{e}^{2\pi k i/n}.$$

SOLUTION: (a) Just as we saw in class on 9/6, in proving that $\sum_{d|n} \varphi(d) = n$ (Thm. 2.2), the set $\{k/n \colon 1 \le k \le n\}$ equals the disjoint union $\cup_{d|n}\{k/d \colon 1 \le k \le d; (k,d) = 1\}$. So

$$F(n) = \sum_{d|n} \sum_{\substack{k=1 \\ (k,d)=1}}^{d} f\left(\frac{k}{d}\right) = \sum_{d|n} F^*(d) = u * F^*(n)$$

where $u(n) = 1$ for all n. So, by Möbius inversion, $F^* = u^{-1} * F = \mu * F$.

(b) follows from **(a)** with $f(x) = \mathrm{e}^{2\pi i x}$. Indeed, in this case,

$$\begin{aligned}\sum_{\substack{k=1 \\ (k,n)=1}}^{n} \mathrm{e}^{2\pi k i/n} &= \sum_{\substack{k=1 \\ (k,n)=1}}^{n} f\left(\frac{k}{n}\right) = F^*(n) = \mu * F(n) = \sum_{d|n} \mu(d) \sum_{k=1}^{n} f\left(\frac{kd}{n}\right) \\ &= \sum_{d|n} \mu(d) \sum_{k=1}^{n} e^{2\pi i k d/n} = \sum_{d|n} \mu(d) \sum_{k=1}^{n} \left(e^{2\pi i d/n}\right)^k = \sum_{d|n} \mu(d) \cdot \begin{cases} n & \text{if } d = n, \\ 0 & \text{if } d < n, \end{cases} \\ &= \mu(n),\end{aligned}$$

the next-to-last equality because

$$\sum_{k=1}^{n} a^k = \begin{cases} n & \text{if } a = 1, \\ \frac{a(a^n-1)}{a-1} & \text{if } a \ne 1, \end{cases}$$

and because $e^{2\pi i d} = 1$ for d an integer.

Exercise 29. Prove that there is a multiplicative arithmetic function g such that

$$\sum_{k=1}^{n} f((k,n)) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right),$$

for every arithmetic function f ((k,n) denotes the gcd). Deduce that

$$\sum_{k=1}^{n} (k,n)\mu((k,n)) = \mu(n).$$

SOLUTION: Since (k,n) is a divisor of n,

$$\sum_{k=1}^{n} f((k,n)) = \sum_{d|n} \sum_{\substack{k=1 \\ (k,n)=d}}^{n} f(d) = \sum_{d|n} f(d) \sum_{\substack{k=1 \\ (k,n)=d}}^{n} 1.$$

Now $(k,n) = d$, for $1 \le k \le n$, iff $k = dr$ where $1 \le r \le n/d$ and $(r, n/d) = 1$. There are $\varphi(n/d)$ such k's, so

$$\sum_{k=1}^{n} f((k,n)) = \sum_{d|n} f(d) \varphi\left(\frac{n}{d}\right).$$

This implies that

$$\sum_{k=1}^{n} (k,n)\mu((k,n)) = \sum_{d|n} d\mu(d)\varphi\left(\frac{n}{d}\right) = n \sum_{d|n} \mu(d) \frac{\varphi(n/d)}{n/d} = \mu(n);$$

the latter equality follows from Mobius inversion and the fact that

$$\frac{\varphi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Part II. Let $\delta(n)$ denote the number of positive divisors of n.

(1) (a) Express $\delta(n)$ in the form

$$\sum_{d|n} f(n)$$

for an appropriate function f.

(b) Prove that

$$\sum_{d|n} \mu(d)\delta(n/d) = 1.$$

(c) Prove that

$$\sum_{d|n} \log d = \frac{\delta(n)}{2} \log n.$$

(Here and throughout, log denotes the natural logarithm.) Note: this does not depend on part (a) or (b) of this problem.

(d) Using parts (a,b,c) above, prove that

$$\log n = -\sum_{d|n} \mu(d)\delta(n/d) \log d.$$

SOLUTION: (a)

$$\delta(n) = \sum_{d|n} u(n)$$

where $u(n) = 1$ for all n.

(b) By part (a), $\delta = u * u$, so by Möbius inversion, $u = \mu * \delta$, which is the desired result.

(c) Note that, for any divisor d of n, $\log n = \log(d \cdot (n/d)) = \log d + \log(n/d)$. So

$$\begin{aligned}\delta(n)\log(n) = \log(n)\sum_{d|n} 1 = \sum_{d|n} \log(n) = \sum_{d|n} \big(\log d + \log(n/d)\big) = \sum_{d|n} \log d + \sum_{d|n} \log(n/d) \\ = \sum_{d|n} \log d + \sum_{d'|n} \log d' = 2\sum_{d|n} \log d,\end{aligned}$$

(for the second-to-last equality, we put $d' = nd$), from which the result follows.

(d). Let $\ell(n) = \log n$ and $\psi(n) = \frac{1}{2}\delta(n)\log n$. By part (c) above, we have $\ell * u = \psi$, so by Möbius inversion, we have $\ell = \mu * \psi$, meaning

$$\begin{aligned}\log n &= \sum_{d|n} \mu(d)\psi(n/d) = \frac{1}{2}\sum_{d|n} \mu(d)\delta(n/d)\log(n/d) \\ &= \frac{1}{2}\left(\sum_{d|n} \mu(d)\delta(n/d)\log n - \sum_{d|n} \mu(d)\delta(n/d)\log d\right) \\ &= \frac{1}{2}\left(\log n \sum_{d|n} \mu(d)\delta(n/d) - \sum_{d|n} \mu(d)\delta(n/d)\log d\right) \\ &= \frac{1}{2}\log n - \frac{1}{2}\sum_{d|n} \mu(d)\delta(n/d)\log d,\end{aligned}$$

the last step by part (b). The result follows immediately.

(2) (a) Express δ in the form $\delta = u * u$ for an appropriate multiplicative function u.

(b) Use the previous part of this problem to conclude that δ is multiplicative.

(c) Prove that

$$\sum_{d|n} \delta(d)^3 = \left(\sum_{d|n} \delta(d)\right)^2.$$

Hint: it suffices to show that this is true for n a power of a prime. (Explain why.)

SOLUTION: (a) We saw above that $\delta = u * u$ for u the unit function ($u(n) = 1\ \forall n$).

(b) Certainly u is multiplicative, so δ is by Theorem 2.14.

(c) Since δ is multiplicative, so is δ^3, and thus so is $\sum_{d|n} \delta^3(d) = \delta^3 * u(n)$. Similarly, $\sum_{d|n} \delta(d) = \delta * u(n)$ is, and therefore so is $(\sum_{d|n} \delta(d))^2$. So it suffices to show that the desired identity holds for prime powers.

But, for p prime and $k = 0, 1, 2, \ldots$,

$$\sum_{d|p^k} \delta^3(d) = \delta^3(1) + \delta^3(p) + \delta^3(p^2) + \cdots + \delta^3(p^k) = 1 + 2^3 + 3^3 + \cdots + (k+1)^3 = \sum_{i=0}^{k+1} i^3$$

$$= \frac{(k+1)^2(k+2)^2}{4} = \left(\sum_{i=0}^{k+1} i\right)^2 = \left(\delta(1) + \delta(p) + \delta(p^2) + \cdots + \delta(p^k)\right)^2 = \left(\sum_{d|p^k} \delta(d)\right)^2,$$

as required.

Part III. 1. Show that $\mathcal{A} = \{\text{arithmetic functions}\}$ is a commutative ring with unity, under addition and Dirichlet multiplication.

SOLUTION: It's clear that $\mathcal{A}$ is closed under addition of functions, and that addition in $\mathcal{A}$ is commutative, associative, has a zero element, given by $0(n) = 0$ for all n, and has additive inverses.

Further, $\mathcal{A}$ is clearly closed under the Dirichlet product, which is commutative and associative by Theorem 2.6, and which clearly distributes over addition. Also, $\mathcal{A}$ has multiplicative identity given by I ($I(n) = [1/n]$ for all n). That should be it, right?

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2. Show that $\mathcal{A}$ is an integral domain. Hint: given a nonzero arithmetic function f, let

$$z_f = \min\{n \in \mathbb{Z}_+ | f(n) \neq 0\}.$$

SOLUTION: Assume that f and g asre nonzero. Then for z_f and z_g as described,

$$f * g(z_f z_g) = \sum_{d|z_f z_g} f(d)g(z_f z_g/d) = \sum_{\substack{d|z_f z_g \\ d \geq z_f}} f(d)g(z_f z_g/d),$$

since $f(d) = 0$ for $d < z_f$. The first term in the sum on the right equals $f(z_f)g(z_g)$, which is nonzero by definition of z_f and z_g. All subsequent terms are zero, since

$$d > z_f \Rightarrow z_f z_g/d < z_g \Rightarrow g(z_f z_g/d) = 0.$$

So $f * g(z_f z_g) = f(z_f)g(z_g)$, which is nonzero by definition of z_f and z_g, so $f * g$ is nonzero.