More on θ^4.

Let $z \in \mathfrak{h}$. Define

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z},$$

and, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a real, 2×2 matrix, define

$$\gamma z = \frac{az+b}{cz+d}, \qquad j(\gamma, z) = cz+d.$$

We've seen that, for $\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ or $\gamma = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$,

$$\theta^4(\gamma z) = j(\gamma, z)^2 \theta^4(z) \qquad (*)$$

$\forall z \in \mathfrak{h}$.

We have:

Lemma

If $(*)$ holds for $\gamma = \gamma_1$ and $\gamma = \gamma_2$, where

$\gamma_1, \gamma_2 \in SL(2, \mathbb{Z}) = \{2\times 2$ integer matrices of determinant $1\}$,

then $(*)$ holds for any γ in the subgroup of $SL(2,\mathbb{Z})$ generated by $\pm\gamma_1$ and $\pm\gamma_2$.

Proof.

First note that, if $(*)$ holds for $\gamma \in GL(2, \mathbb{R})$, then $(*)$ holds for $-\gamma$, since

$$\left(-\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) z = \frac{-az-b}{-cz-d} = \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z, \text{ and}$$

$$j\left(-\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right)^2 = (-cz-d)^2 = (cz+d)^2 = j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right)^2.$$

Next we show that, if $(*)$ holds for two matrices, then it holds for their product. That is, assume $(*)$ holds for $\gamma=\gamma_1$ and $\gamma=\gamma_2$ $(\gamma_1,\gamma_2\in SL(2,\mathbb{Z}))$. We wish to show that

$$\theta^4(\gamma_1\gamma_2 z)=j(\gamma_1\gamma_2,z)^2\theta^4(z). \qquad (1)$$

To do so we note that, by $(*)$ applied twice, first with $\gamma=\gamma_1$ and then with $\gamma=\gamma_2$,

$$\begin{aligned}\theta^4(\gamma_1\gamma_2 z) &= \theta^4(\gamma_1(\gamma_2 z))\\ &= j(\gamma_1,\gamma_2 z)^2\theta^4(\gamma_2 z)\\ &= j(\gamma_1,\gamma_2 z)^2 j(\gamma_2,z)^2\theta^4(z). \qquad (2)\end{aligned}$$

Comparing (1) and (2), it suffices to show that

$$j(\gamma_1\gamma_2,z)=j(\gamma_1,\gamma_2 z)j(\gamma_2,z). \qquad (3)$$

Let $\gamma_1=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\gamma_2=\begin{pmatrix}p&q\\r&s\end{pmatrix}$.

Then $\gamma_1\gamma_2=\begin{pmatrix}ap+br & aq+bs\\ cp+dr & cq+ds\end{pmatrix}$,

so

$$j(\gamma_1\gamma_2,z)=(cp+dr)z+(cq+ds).$$

On the other hand, $\gamma_2 z=\dfrac{pz+q}{rz+s}$, so

$$j(\gamma_1,\gamma_2 z)j(\gamma_2,z)=\left[c\left(\frac{pz+q}{rz+s}\right)+d\right](rz+s)$$

$$= c(pz+q) + d(rz+s)$$

$$= (cp+dr)z + cq+ds = j(\gamma_1\gamma_2, z),$$

as desired.

Finally we need to show that, if $(*)$ is true for $\gamma \in SL(2,\mathbb{Z})$, then it's true for γ^{-1}.

To do so, assume $(*)$, and replace z by $\gamma^{-1}z$ to get

$$\theta^4(z) = j(\gamma, \gamma^{-1}z)^2\theta^4(\gamma^{-1}z),$$

or

$$\theta^4(\gamma^{-1}z) = j(\gamma,\gamma^{-1}z)^{-2}\theta^4(z).$$

So it suffices to show that

$$j(\gamma,\gamma^{-1}z)^{-1} = j(\gamma^{-1}, z),$$

or $j(\gamma,\gamma^{-1}z)j(\gamma^{-1},z) = 1$.

But by (3) with $\gamma_1 = \gamma$ and $\gamma_2 = \gamma^{-1}$;

$$j(\gamma,\gamma^{-1}z)j(\gamma^{-1}, z) = j(\gamma\gamma^{-1}, z) = j\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, z\right)$$

$= 0z+1 = 1$, and we're done. $\square$

<u>Corollary</u>

Let $\Gamma_0(4) = \{\text{matrices } \begin{pmatrix}a&b\\c&d\end{pmatrix} : ad-bc=1 \text{ and } 4|c\}$. Then

$$\theta^4(\gamma z) = j(\gamma, z)^2\theta^4(z)$$

$\forall z \in \mathfrak{h}$, $\gamma \in \Gamma_0(4)$.

Proof.

One may show that $\Gamma_0(4)$ is the group generated by

$$\left\{ \pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ \pm\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right\}.$$

Because, again, $(*)$ holds for

$$\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},$$

the result follows from the Lemma. $\square$

Definition.

A function $f: \mathfrak{h} \to \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$ for a subgroup Γ of $SL(2,\mathbb{Z})$ if:

(a) f is holomorphic on $\mathfrak{h}$;

(b) $f(\gamma z) = j(\gamma, z)^k f(z) \quad \forall \gamma \in \Gamma,\ z \in \mathfrak{h}$;

(c) f is bounded in z as $\operatorname{Im} z \to \infty$.

The vector space of modular forms of weight k for Γ is denoted $\mathcal{M}_k(\Gamma)$.

We have:

Theorem

$\Theta^4 \in \mathcal{M}_2(\Gamma_0(4))$.

Proof.

(a) Θ, and thus Θ^4, is holomorphic on $\mathfrak{h}$ by Prop. 1 of last time.

(b) Θ^4 satisfies (b), with $k=2$ and $\Gamma = \Gamma_0(4)$, by the above Corollary.

(c) To see that Θ, and thus Θ^4, is bounded as $\mathrm{Im}(z) \to \infty$, note that, for $t > 1$,

$$\left| \sum_{n \in \mathbb{Z}} e^{-2\pi i n^2(\sigma + it)} \right| \leq 1 + 2\sum_{n=1}^{\infty} e^{-2\pi n^2 t}$$

$$= 1 + 2\sum_{n=1}^{\infty} e^{-2\pi n^2(t-1)} e^{-2\pi n^2}$$

$$\leq 1 + 2e^{-2\pi(t-1)} \sum_{n=1}^{\infty} e^{-2\pi n}.$$

The series is a convergent geometric series, and clearly $e^{-2\pi(t-1)} \to 0$ as $t \to \infty$.

$\square$