Jacobi's theta function and sums of four squares.

Goals:

I) To interpret a variant of Jacobi's theta function $\Theta(t)$ as a "modular form;"

II) To use this to study sums of four squares.

Part I: The function $\Theta(z)$.

Define the <u>upper half plane</u> $\mathfrak{h}$ by

$$\mathfrak{h} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}.$$

Throughout, for $z \in \mathfrak{h}$, we write $z = \sigma + it$ with $\sigma, t \in \mathbb{R}$, $t > 0$.

Now define $\Theta : \mathfrak{h} \to \mathbb{C}$ by

$$\Theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}.$$

We have:

Proposition 1.
Θ is analytic on $\mathfrak{h}$.

Proof
This follows essentially from the fact that

$$\left| e^{2\pi i n^2 z} \right| = \left| e^{2\pi i n^2 (\sigma + it)} \right| = e^{-2\pi n^2 t},$$

which implies that the series for $\Theta(z)$ converges uniformly on compact subsets of $\mathfrak{h}$. $\square$

Next:

<u>Proposition 2.</u> For $z \in \mathfrak{h}$,

(a) $\theta(z+1) = \theta(z)$,

(b) $\theta\left(\dfrac{-1}{4z}\right) = \sqrt{-2iz}\,\theta(z)$.

<u>Proof.</u>

(a)

$$\theta(z+1) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2(z+1)}$$
$$= \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} \cdot e^{2\pi i n^2}$$
$$= \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z} = \theta(z).$$

(b) By analytic continuation, it suffices to prove this for z imaginary - that is, it suffices to show that

$$\theta\left(\frac{-1}{4(0+it)}\right) = \sqrt{-2i(0+it)}\,\theta(0+it)$$

or

$$\theta(i/4t) = \sqrt{2t}\,\theta(it).$$

But note that, in terms of Jacobi's theta function $\Theta(t)$,

$$\theta(i/4t) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \cdot i/(4t)} = \sum_{n\in\mathbb{Z}} e^{-\pi n^2/(2t)}$$

$$= \Theta(1/(2t)),$$

which, by the Jacobi identity, equals

$$\sqrt{2t}\,\Theta(2t) = \sqrt{2t}\sum_{n\in\mathbb{Z}} e^{-\pi n^2\cdot 2t}$$

$$= \sqrt{2t}\sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \cdot it} = \sqrt{2t}\,\theta(it),$$

as claimed. $\square$

We shift our focus to θ^4. Note that Prop. 2(b) implies

$$\theta^4(-\tfrac{1}{4z}) = (2iz)^2\theta^4(z)$$
$$= -4z^2\theta^4(z). \qquad (*)$$

We have:

<u>Corollary 1.</u> For $z\in\mathfrak{h}$,

(a) $\theta^4(z-1) = \theta^4(z)$;

(b) $\theta^4(z/(4z+1)) = (4z+1)^2\theta^4(z)$.

<u>Proof</u>

(a) Immediate from Prop. 2(a).

(b) Note that

$$\frac{z}{4z+1} = \frac{-1}{4(-1/(4z)-1)} = \frac{-1}{4w},$$

where $w = v-1$ and $v = -1/4z$.

So by part (a) of this corollary and by $(*)$,

$$\theta^4(z/(4z+1)) = \theta^4(-1/4\omega)$$

$$= -4\omega^2\theta^4(\omega) = -4(v-1)^2\theta^4(v-1)$$

$$= -4(v-1)^2\theta^4(v) = -4(v-1)^2\theta^4(-1/4z)$$

$$= -4(v-1)^2(-4z^2)\theta^4(z)$$

$$= 16\left(\frac{-1}{4z}-1\right)^2 z^2\,\theta^4(z)$$

$$= 16\left(\frac{-1-4z}{4z}\right)^2 z^2\theta^4(z) = (4z+1)^2\theta^4(z).$$

$\square$

Remark: Corollary 1 says that, if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is <u>either</u> $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, then

$$\theta^4\left(\frac{az+b}{cz+d}\right) = (cz+d)^2\theta(z).$$

More next time.