Lemma Γ and Theorem 5.

Lemma Γ.

The Gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} \qquad (\mathrm{Re}(s) > 0)$$

has the following properties:

(a) Analytic continuation to $D = \mathbb{C} - \{0, -1, -2, \ldots\}$, with a simple pole at each nonpositive integer $-n$, with

$$\mathrm{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!} \qquad (n \in \mathbb{Z}_{\geq 0}).$$

(b) $\Gamma(s)$ is never zero.

Proof.

(a) By the integral formula above, $\Gamma(s)$ is analytic for $\mathrm{Re}(s) > 0$.

We've seen that

$$\Gamma(s+1) = s\Gamma(s) \quad \text{for such } s.$$

By induction, we find that

$$\Gamma(s+n+1) = (s+n)(s+n-1)\cdots(s+1)s\,\Gamma(s)$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\mathrm{Re}(s) > 0$. Rearranging gives

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n-1)(s+n)}. \qquad (*)$$

The right side is defined and analytic

for $\mathrm{Re}(s) \geq -n-1$ and $s \neq 0, -1, -2, \ldots, -n$, so it defines an analytic continuation of $\Gamma(s)$ to that domain. This holds for any $n \in \mathbb{Z}_{\geq 0}$, which gives the continuation of $\Gamma(s)$ to Ω.

Moreover, $(*)$ gives, for $n \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{Res}_{s=-n} \Gamma(s) = \mathrm{Res}_{s=-n} \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n-1)(s+n)}$$

$$= \lim_{s \to -n} \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n-1)}$$

$$= \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{1}{(-1)^n (n)(n-1)\cdots 1}$$

$$= \frac{(-1)^n}{n!}.$$

(b) It suffices to show that $\Gamma(s) \neq 0$ for $\mathrm{Re}(s) > 0$, because this and $(*)$ will imply that $\Gamma(s) \neq 0$ for $\mathrm{Re}(s) > -n-1$, for any $n \in \mathbb{Z}_{\geq 0}$.

So suppose $\mathrm{Re}(s) > 0$ and $\Gamma(s) = 0$. The zeroes of an analytic function are isolated, so we can choose $\varepsilon > 0$ such that:

(a) $\Gamma(s-\varepsilon) \neq 0$,
(b) $\mathrm{Re}(s-\varepsilon) > 0$.

By HW7 Exercise 1,

$$\Gamma(s-\varepsilon)\Gamma(\varepsilon) = \Gamma(s) \int_0^1 u^{s-\varepsilon-1} (1-u)^{\varepsilon-1} \, du.$$

The right side is zero by assumption. But $\Gamma(s-\varepsilon) \neq 0$ by choice of ε, and $\Gamma(\varepsilon) \neq 0$

for $\varepsilon > 0$ by the integral definition of $\Gamma(\varepsilon)$.

Contradiction. So $\Gamma(s) \neq 0$. $\square$

<u>Theorem 5</u>: analytic continuation and functional equation for $\zeta(s)$.

(a) The zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\mathrm{Re}(s) > 1)$$

has analytic continuation to $\mathbb{C} - \{1\}$, with a simple pole of residue 1 at $s = 1$.

(b) $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has analytic continuation to $\mathbb{C} - \{0,1\}$, has a simple pole of residue ± 1 at $s = \frac{1}{2}(1 \pm 1)$, and satisfies

$$\Lambda(s) = \Lambda(1-s).$$

<u>Proof</u>. For $\mathrm{Re}(s) > 1$,

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\sum_{n=1}^{\infty} n^{-s}$$

$$= \sum_{n=1}^{\infty} \pi^{-s/2} n^{-s} \int_0^{\infty} e^{-t} t^{s/2} \frac{dt}{t}$$

$$t \to t\pi n^2 \quad = \sum_{n=1}^{\infty} \pi^{-s/2} n^{-s} (\pi n^2)^{s/2} \int_0^{\infty} e^{-t\pi n^2} t^{s/2} \frac{dt}{t}$$

$$= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-t\pi n^2} t^{s/2} \frac{dt}{t} = \int_0^{\infty} \Big(\sum_{n=1}^{\infty} e^{-t\pi n^2}\Big) t^{s/2} \frac{dt}{t}$$

$$= \int_0^{\infty} \frac{1}{2}\Big(\sum_{n\in\mathbb{Z}} e^{-t\pi n^2} - 1\Big) t^{s/2} \frac{dt}{t}$$

$$= \int_0^\infty \tfrac{1}{2}(\Theta(t)-1)\, t^{s/2}\, \frac{dt}{t}$$

$$= \int_1^\infty \tfrac{1}{2}(\Theta(t)-1)t^{s/2}\frac{dt}{t} + \int_0^1 \tfrac{1}{2}(\Theta(t)-1)t^{s/2}\frac{dt}{t}$$

$t \to 1/t$ in second integral; combine integrals

$$= \int_1^\infty \tfrac{1}{2}\left[(\Theta(t)-1)t^{s/2} + (\Theta(1/t)-1)t^{-s/2}\right]\frac{dt}{t}$$

Jacobi identity

$$= \int_1^\infty \tfrac{1}{2}\left[(\Theta(t)-1)t^{s/2} + (t^{1/2}\Theta(t)-1)t^{-s/2}\right]\frac{dt}{t}$$

$$= \tfrac{1}{2}\int_1^\infty (\Theta(t)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t}$$
$$+ \tfrac{1}{2}\int_1^\infty \left(t^{(1-s)/2} - t^{-s/2}\right)\frac{dt}{t}$$

$$= \tfrac{1}{2}\int_1^\infty (\Theta(t)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} + \frac{1}{s-1} - \frac{1}{s}.$$

The remaining integral is entire in s, by the rapid decay of $\Theta(t)-1$ as $t\to\infty$. This integral is also invariant under $s \to 1-s$, as is $1/(s-1) - 1/s$. Theorem 5(b) follows readily.

To get part (a), use the above to write

$$\zeta(s) = \frac{\Lambda(s)}{\pi^{-s/2}\Gamma(s/2)}$$

$$= \frac{\pi^{s/2}}{2\Gamma(s/2)}\int_1^\infty (\Theta(t)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t}$$
$$+ \frac{\pi^{s/2}}{(s-1)\Gamma(s/2)} - \frac{\pi^{s/2}}{s\Gamma(s/2)}.$$

The first term on the right is entire by Lemma Γ(b). The third term equals

$$-\frac{\pi^{s/2}}{(2\cdot s/2)\Gamma(s/2)} = -\frac{\pi^{s/2}}{2\Gamma(s/2+1)},$$

which is entire for the same reason. Finally, the second term on the right has residue

$$\mathrm{Res}_{s=1} \frac{\pi^{s/2}}{(s-1)\Gamma(s)} = \lim_{s\to 1} \frac{\pi^{s/2}}{\Gamma(s)} = \frac{\pi^{1/2}}{\Gamma(1/2)} = 1$$

by HW 7 #2, and we're done. $\square$