Some more $\mathfrak{Z}$ lemmas.

For this, we'll need to recall the definition of the Fourier transform $\hat{f}$ of a suitably nice function $f : \mathbb{R} \to \mathbb{C}$:

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx \qquad (s \in \mathbb{R}).$$

We have:

Lemma β. If $G(x) = e^{-\pi x^2}$, then $\hat{G} = G$.

Proof

Note that $G(0) = 1$, and that

$$G'(x) = -2\pi x e^{-\pi x^2} = -2\pi x G(x).$$

So G satisfies the initial value problem

$$f'(x) = -2\pi x f(x), \qquad f(0) = 1. \qquad (*)$$

By elementary theory of differential eq'ns, $(*)$ has a unique solution, so if we can show that $\hat{G}$ also satisfies $(*)$, we'll be done.

We have

$$\hat{G}(s) = \int_{-\infty}^{\infty} G(x) e^{-2\pi i s x}\, dx,$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i s x}\, dx,$$

so

$$(\hat{G})'(s) = -2\pi i \int_{-\infty}^{\infty} x e^{-\pi x^2} e^{-2\pi i s x}\, dx.$$

Integrate by parts with $u = ie^{-2\pi i s x}$ and $dv = -2\pi x e^{-\pi x^2}dx$. We get $du = 2\pi s e^{-2\pi i s x}dx$ and $v = e^{-\pi x^2}$, so

$$(\hat{G})'(s) = ie^{-\pi x^2}e^{-2\pi i s x}\Big|_{-\infty}^{\infty}$$

$$+ 2\pi s \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i s x}dx$$

$$= 0 + 2\pi s \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i s x}dt$$

$$= 2\pi s \hat{G}(s),$$

so $\hat{G}$ satisfies the differential equation in $(*)$. To see that it satisfies the initial condition there, we compute

$$\hat{G}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i (0) x}dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1$$

by Lemma α, and we're done.

A corollary:

<u>Lemma γ.</u>

If $t > 0$ and $G_t(x) = e^{-\pi t x^2}$, then $\hat{G_t}(s) = t^{-1/2} e^{-\pi s^2/t}$.

<u>Proof</u> Let $G(x) = \hat{G}(x) = e^{-\pi x^2}$ as above. We have

$$\hat{G}_t(s) = \int_{-\infty}^{\infty} e^{-\pi t x^2} e^{-2\pi i s x}\, dx$$

$x \to x/\sqrt{t}$

$$= t^{-1/2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i (s/\sqrt{t}) x}\, dx$$

$$= t^{-1/2} \hat{G}(s/\sqrt{t}) = t^{-1/2} G(s/\sqrt{t})$$

$$= t^{-1/2} e^{-\pi (s/\sqrt{t})^2} = t^{-1/2} e^{-\pi s^2/t}. \qquad \square$$

For the next lemma we recall that, if g is a suitably nice function on $\mathbb{R}$ and g has period 1, then g has a Fourier series

$$g(x) = \sum_{n=-\infty}^{\infty} c_n(g) e^{2\pi i n x},$$

where

$$c_n(g) = \int_0^1 g(y) e^{-2\pi i n y}\, dy.$$

Consequence:

<u>Lemma 5</u> (Poisson summation)

If f is a function on $\mathbb{R}$ such that both $f(x)$ and $\hat{f}(x)$ are of suitably rapid decay as $x \to \pm\infty$, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

<u>Proof</u> For such f, define

$$g(x) = \sum_{n\in\mathbb{Z}} f(x+n). \qquad (A)$$

Then g has period 1, so it has a Fourier series

$$g(x) = \sum_{n\in\mathbb{Z}} c_n(g) e^{2\pi i n x}$$

$$= \sum_{n\in\mathbb{Z}} \left[\int_0^1 g(y) e^{-2\pi i n y}\, dy\right] e^{2\pi i n x} dx$$

$$= \sum_{n\in\mathbb{Z}} \left[\int_0^1 \sum_{m\in\mathbb{Z}} f(y+m) e^{-2\pi i n y}\, dy\right] e^{2\pi i n x} dx$$

$$= \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} \int_0^1 f(y+m) e^{-2\pi i n y}\, dy\right) e^{2\pi i n x} dx$$

$y \to y - m$

$$= \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} \int_m^{m+1} f(y) e^{-2\pi i n (y-m)}\, dy\right) e^{2\pi i n x}$$

$e^{2\pi i n m} = 1$

$$= \sum_{n\in\mathbb{Z}} \left(\int_{-\infty}^{\infty} f(y) e^{-2\pi i n y}\, dy\right) e^{2\pi i n x}.$$

$$= \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x}. \qquad (B)$$

Equating (A) and (B), and letting $x=0$, gives the result. ∎

Finally:

Lemma E.

If the "Jacobi theta function" $\Theta(t)$ is defined by

$$\Theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi t n^2},$$

then we have the "Jacobi identity"

$$\Theta(t) = t^{-1/2}\,\Theta(1/t).$$

<u>Proof.</u>
Note that

$$\Theta(t) = \sum_{n\in\mathbb{Z}} G_t(n)$$

with $G_t(x) = e^{-\pi t x^2}$, as above. By Lemma 5, then,

$$\Theta(t) = \sum_{n\in\mathbb{Z}} \widehat{G_t}(n),$$

so by Lemma 6,

$$\Theta(t) = \sum_{n\in\mathbb{Z}} t^{-1/2} e^{-\pi n^2/t}$$

$$= t^{-1/2}\,\Theta(1/t). \quad \square$$