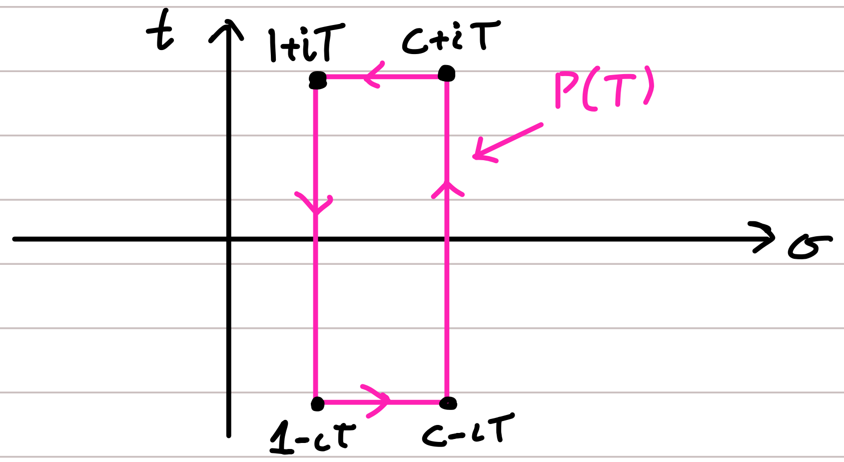
PNT STEP 4, continued.




Recall: to prove PNT, it suffices to show that

(A) For $x \geq 1$ and for any $c > 1$,

$$\lim_{T\to\infty} \int_{1-iT}^{c-iT} x^{s-1} h(s)\, ds = 0,$$

and the same holds for the integral over $[1+iT, c+iT]$; and

(B) $$\lim_{x\to\infty} \int_{-\infty}^{\infty} h(1+it)\, e^{it\log x}\, dt = 0.$$

Here, again, $h(s) = \dfrac{1}{s(s+1)}\left(-\dfrac{\zeta'(s)}{\zeta(s)} - \dfrac{1}{s-1}\right).$

Today, we prove (A).

Theorem 13.7. $\exists\, K > 0$ such that

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq K \log^9 t$$

for $t$ sufficiently large and $1 \le \sigma \le 2$.

<u>Proof</u>. Suppose $t \geq c$ and $\sigma > 1$. By Thm. 13.5,

$$|\zeta(\sigma+it)| \geq \zeta(\sigma)^{-3/4} |\zeta(\sigma+2it)|^{-1/4}. \qquad (*)$$

Now $\zeta(s)$ has a simple pole as $s=1$, so $(\sigma-1)\zeta(\sigma)$ is bounded near $\sigma=1$, say $|(\sigma-1)\zeta(\sigma)| \le N$ for $1 \le \sigma \le 1+\varepsilon$. But for $1+\varepsilon \le \sigma \le 2$, $|(\sigma-1)\zeta(\sigma)| \le 2\zeta(1+\varepsilon)$. So $(\sigma-1)\zeta(\sigma)$ is bounded for $1 \le \sigma \le 2$. So $\exists B>0$ such that, for $1<\sigma \le 2$, $\zeta(\sigma)^{-3/4} \geq B(\sigma-1)^{3/4}$. So by $(*)$,

$$|\zeta(\sigma+it)| \geq B(\sigma-1)^{3/4} |\zeta(\sigma+2it)|^{-1/4}$$

for $1<\sigma\le 2$. But by Thm. 13.4, $|\zeta(\sigma+2it)| = O(\log 2t) = O(\log t)$ for $1<\sigma\le 2$ and $t \geq c$. So $\exists A>0$ such that, for such $\sigma$ and $t$,

$$|\zeta(\sigma+it)| \geq A(\sigma-1)^{3/4} \log^{-1/4} t. \qquad (1)$$

Clearly this is true for $\sigma=1$ as well.

Now suppose $1<\alpha<2$ and $t \geq c$. If $1 \le \sigma \le \alpha$, then

$$|\zeta(\alpha+it) - \zeta(\sigma+it)| = \left| \int_\sigma^\alpha \zeta'(u+it)\,du \right|$$

$$\le \int_\sigma^\alpha |\zeta'(u+it)|\,du$$

which, by Thm. 13.4, is, for some $M>0$,

$$\le (\alpha-\sigma) M \log^2 t \le (\alpha-1) M \log^2 t.$$

So, by the triangle inequality and by (1),

$$|\zeta(\sigma+it)| \geq |\zeta(\alpha+it)| - |\zeta(\alpha+it)-\zeta(\sigma+it)|$$

$$\geq |\zeta(\alpha+it)| - (\alpha-1) M \log^2 t$$

$$\geq A(\alpha-1)^{3/4} \log^{-1/4} t - (\alpha-1) M \log^2 t. \quad (2)$$

But (2) is also true for $\alpha \le \sigma \le 2$, because then $(\sigma-1)^{3/4} \geq (\alpha-1)^{3/4}$, so by (1),

$$|\zeta(\sigma+it)| \geq A(\alpha-1)^{3/4} \log^{-1/4} t$$
$$\geq A(\alpha-1)^{3/4} \log^{-1/4} t - (\alpha-1) M \log^2 t.$$

In other words, (2) holds for all $1<\alpha<2$, $1 \le \sigma \le 2$, and $t \geq e$.

Choose $\alpha$ to depend on $t$:

$$\alpha = 1 + \left(\frac{A}{2M}\right)^4 \log^{-9} t.$$

Note $\alpha>1$; also $\alpha<2$ if $t$ is large enough, say $t \geq t_0$. So for such $t$ and for $1 \le \sigma \le 2$, by (2),

$$|\zeta(\sigma+it)| \geq \frac{A^4}{(2M)^3} \log^{-27/4} t \cdot \log^{-1/4} t - \frac{A^4}{2^4 M^3} \log^{2-9} t$$

$$= \frac{A^4}{2^4 M^3} \log^{-7} t.$$

But also by Thm. 13.4, $|\zeta'(\sigma+it)| \le M \log^2 t$ for such $\sigma$ and $t$, for suitably chosen $M > 0$. But then

$$\left| \frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} \right| \le M \log^2 t \cdot \frac{2^4 M^3}{A^4} \log^7 t$$
$$= \left(\frac{2M}{A}\right)^4 \log^9 t$$

for $1 \le \sigma \le 2$ and $t \ge t_0$, and we're done. $\square$

Corollary

(A) above is true; that is, for $c > 1$ and $x \ge 1$,

$$\lim_{T\to\infty} \int_{1+iT}^{c+iT} x^{s-1} h(s)\, ds = 0,$$

and the same is true with $-T$ in place of $T$.

Proof. We first consider the integral over $[1+iT, c+iT]$. Write $s = \sigma + iT$, and note that

$$|s+j|^{-1} \le T^{-1} \text{ for } j = -1, 0, \text{ or } 1.$$

So, for $T$ sufficiently large, and for $1 < c \le 2$ and $x \ge 1$, we have, by Thm. 13.7,

$$\left|\int_{1+iT}^{c+iT} x^{s-1} h(s)\,ds\right|$$

$$\leq \int_1^c \left|x^{\sigma-1+iT}\right| \left|h(\sigma+iT)\right| d\sigma$$

$$\leq (c-1)x^{c-1} \cdot \frac{1}{T^2}\left(K\log^q T + \frac{1}{T}\right),$$

which approaches $0$ as $T \to \infty$.

The integral over $[1-iT, c-iT]$ is similar, because $h$ is analytic on this path and on the previous one, so

$$|h(\sigma-iT)| = |h(\overline{\sigma+iT})| = |\overline{h(\sigma+iT)}|$$

$$= |h(\sigma+iT)| \text{ for } 1 \leq \sigma \leq c \text{ and } T>0.$$