PNT STEP 4, continued.

Theorem 13.5
If $\sigma > 1$ then

$$\zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geq 1.$$

Proof.
Recall Lemma F of last time: for $\sigma > 1$,

$$\zeta(s) = e^{G(s)} \text{ where}$$

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

Since $p^{-ms} = p^{-m\sigma} p^{-mit} = p^{-m\sigma} e^{-imt\log p}$, then

$$\zeta(s) = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{e^{-imt\log p}}{mp^{m\sigma}} \right).$$

So, because $|e^z| = e^{\mathrm{Re}(z)}$ and $\mathrm{Re}(e^{-imt\log p}) = \cos(mt\log p)$, we have

$$|\zeta(s)| = \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{\cos(mt\log p)}{mp^{m\sigma}} \right).$$

Applying this with $s=\sigma$, $s=\sigma+it$, and $s=\sigma+2it$ respectively gives

$$\zeta^3(\sigma)\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)|$$

$$= \exp\left(\sum_p \sum_{m=1}^{\infty} \frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}} \right). \quad (*)$$

But for any real number θ,

$$3+4\cos\theta+\cos 2\theta = 3+4\cos\theta+2\cos^2\theta-1$$
$$= 2(1+2\cos\theta+\cos^2\theta)$$
$$= 2(1+\cos\theta)^2 \geq 0,$$

so the right side of $(*) = e^A$ where $A \geq 0$, whence $e^A \geq 1$, and we're done. $\square$

Consequently:

<u>Thm. 13.6</u>. $\zeta(1+it) \neq 0$ for t real.

<u>Proof</u>.
$\zeta(s)$ has a pole at $s=1$, so certainly $\zeta(1) \neq 0$.
So suppose $t \neq 0$. Dividing both sides of Thm. 13.5 by $\sigma-1$ gives

$$((\sigma-1)\zeta(\sigma))^3 \left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\zeta(\sigma+2it)| \geq \frac{1}{\sigma-1}, \qquad (**)$$

for $\sigma>1$. Now let $\sigma \to 1^+$. Then

$$((\sigma-1)\zeta(\sigma))^3 \to 1^3 = 1$$

(since ζ has a simple pole at $s=1$), and $\zeta(\sigma+2it) \to \zeta(1+2it)$.

Now were it the case that $\zeta(1+it)=0$, then we'd have

$$\frac{\zeta(\sigma+it)}{\sigma-1} = \frac{\zeta(\sigma+it)-\zeta(1+it)}{(\sigma+it)-(1+it)},$$

which would approach $\zeta'(1+it)$ as $\sigma \to 1^+$. So the left side of $(**)$ would approach

$$|\zeta'(1+it)|^4 \, |\zeta(1+2it)|$$

as $\sigma \to 1^+$, contradicting the fact that the right side $\to +\infty$ as $\sigma \to 1^+$.

So $\zeta(1+it) \neq 0$. $\square$

A consequence:

<u>Thm. 13.8.</u>

$$h(s) = \frac{1}{s(s+1)}\left(\frac{-\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right)$$

is analytic for $\sigma \geq 1$.

<u>Proof.</u>

For $\sigma > 1$, we've seen (cf. Lemma C of 11/1) that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

which is analytic for $\sigma > 1$, e.g. by Corollary $\mathbb{C}$-Σ. Also, by Lemma E of this Monday, ζ and ζ' are holomorphic for $\sigma > 0$ and $s \neq 1$.

Since $\zeta(1+it)\neq 0$ for $t\in\mathbb{R}$, it follows that ζ'/ζ is holomorphic on the same region. So we only need to consider $s=1$.

Since $\zeta(s)$ has a simple pole there, of residue 1, we have

$$\zeta(s) = \frac{f(s)}{s-1},$$

where $f(s)$ is holomorphic and nonzero at $s=1$. But then, near $s=1$,

$$\zeta'(s) = \frac{(s-1)f'(s)-f(s)}{(s-1)^2},$$

so

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{f'(s)}{f(s)} - \frac{1}{s-1},$$

so

$$h(s) = \frac{1}{s(s+1)} \cdot \frac{-f'(s)}{f(s)},$$

so $h(s)$ is analytic at $s=1$. $\square$

Now recall: for $c>1$ and $x\geq 1$,

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(s)\, x^{s-1}\, ds,$$

and the PNT will follow if we can show

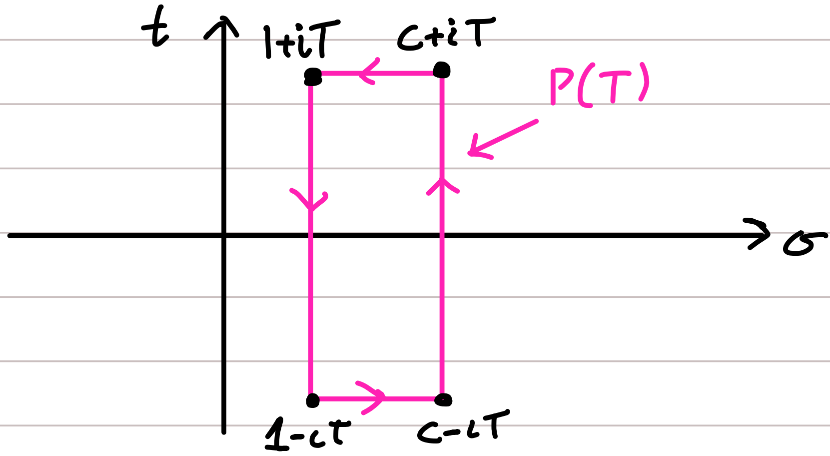
(a) The quantity on the right makes sense, and is unchanged, if we put $c=1$;

(b) With $c=1$, this quantity $\to 0$ as $x \to \infty$.

Regarding part (a) note that, by Thm. 13.8,

$$\frac{1}{2\pi i}\int_{P(T)} h(s)\,x^{s-1}\,ds = 0,$$

where $P(T)$ is as follows, for $c>1$ and $T>0$:



If we can show the integrals over the horizontal portions of $P(T) \to 0$ as $T \to \infty$, we will have proved (a). (Next time.)