STEP 4, continued.

From now on, for $s \in \mathbb{C}$, we write $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$, so $s = \sigma + it$.

<u>Theorem 13.4</u> (upper bounds for $\zeta(s)$ and $\zeta'(s)$).

$\exists M > 0$ such that, for all $t \geq e$ and $\sigma \geq 1$,

$$|\zeta(s)| \leq M \log t \quad \text{and} \quad |\zeta'(s)| \leq M \log^2 t.$$

<u>Proof.</u>

First assume $\sigma \geq 2$ and $t \geq e$. Then

$$|\zeta(s)| = \left|\sum_{n=1}^{\infty} n^{-s}\right| \leq \sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-2}$$

$$= \zeta(2) \leq \zeta(2) \log t,$$

and similarly, since $|\mu(n)| \leq 1 \ \forall n$,

$$|\zeta'(s)| = \left|\sum_{n=1}^{\infty} \mu(n) n^{-s}\right| \leq \zeta(2) \leq \zeta(2) \log^2 t.$$

So take $M = \zeta(2)$.

Now suppose $1 \leq \sigma < 2$ and $t \geq e$. Note first that

$$|s| = |\sigma + it| \leq \sigma + t < 2 + t < t + t = 2t, \text{ and}$$

$$|s-1| = |\sigma + 1 + it| = \sqrt{(\sigma-1)^2 + t^2} \geq t, \text{ so that}$$

$1/|s-1| \le 1/t$. Now by Lemma E of last time, for $N \in \mathbb{Z}_+$ and $1 \le \sigma < 2$,

$$|\zeta(s)| = \left| \sum_{n=1}^{N} 1/n^s - s\int_N^\infty \frac{x-[x]}{x^{s+1}}\,dx + \frac{N^{1-s}}{s-1} \right|$$

$$\le \sum_{n=1}^{N} 1/n^\sigma + 2t\int_N^\infty \frac{dx}{x^{\sigma+1}} + \frac{N^{1-\sigma}}{t}$$

$$\le \sum_{n=1}^{N}\frac{1}{n} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t} \le \sum_{n=1}^{N}\frac{1}{n} + \frac{2t}{N} + \frac{1}{t}.$$

Note that the right hand side is independent of σ.

Now choose $N = [t]$. then $N = O(t)$, $2t/N = O(1)$, and $N/t = o(1)$, and, by Thm. 3.2(a),

$$\sum_{n=1}^{N}\frac{1}{n} = O(\log N) = O(\log t). \quad \text{So}$$

$$|\zeta(s)| = O(\log t) + O(1) + o(1) = O(\log t)$$

(independently of σ), as claimed.

Similarly, again by Lemma E, for $\sigma \ge 2$, $t \ge e$, and $N \in \mathbb{Z}_+$;

$$|\zeta'(s)| \le \left| -\sum_{n=1}^{N}\frac{\log n}{n^s} + s\int_N^\infty \frac{(x-[x])\log x}{x^{s+1}}\,dx \right.$$

$$\left. - \int_N^\infty \frac{x-[x]}{x^{s+1}}\,dx - \frac{N^{1-s}\log N}{s-1} - \frac{N^{1-s}}{(s-1)^2} \right|$$

$$\leq \sum_{n=1}^{N} \frac{\log n}{n} + 2t \int_N^\infty \frac{\log x}{x^{\sigma+1}}\, dx$$

$$+ \int_N^\infty x^{-1-\sigma}\, dx + \frac{\log N}{t} + \frac{1}{t^2} .$$

The sum on the right is $O\left(\log N \sum_{n=1}^{N} \frac{1}{n}\right)$ $= O(\log^2 N)$, by Thm. 3.2(a).

Also, integrating by parts,

$$2t \int_N^\infty \frac{\log x}{x^{\sigma+1}}\, dx = 2t\left(\frac{\log N}{\sigma N^\sigma} + \frac{1}{\sigma^2 N^\sigma}\right).$$

Putting $N = [t]$, the necessary estimates then follow as before. $\square$

For some <u>lower</u> bounds for $\zeta(s)$, we'll need

<u>Lemma F</u>

For $\sigma > 1$, we have

$$\zeta(s) = e^{G(s)},$$

where

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}.$$

<u>Proof</u>

First we rewrite $G(s)$:

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} = \sum_p \sum_{m=1}^{\infty} \frac{\Lambda(p^m)}{\log p^m} (p^m)^{-s}$$

$$= \sum_p \sum_{m=1}^{\infty} \frac{\log p}{m \log p} p^{-ms} = \sum_p \sum_{m=1}^{\infty} \frac{1}{m p^{ms}} .$$

Also, for $|z| < 1$,

$$-\log(1-z) = \sum_{m=1}^{\infty} \frac{z^m}{m} ;$$

putting $z = p^{-s}$ gives

$$-\log(1-p^{-s}) = \sum_{m=1}^{\infty} \frac{1}{m p^{ms}} ;$$

Summing over all primes p gives

$$-\sum_p \log(1-p^{-s}) = \sum_p \sum_{m=1}^{\infty} \frac{1}{m p^{ms}} = G(s).$$

Exponentiate both sides to get

$$\prod_p (1-p^{-s})^{-1} = e^{G(s)} .$$

The left side is $\zeta(s)$ by Lemma E(c) of last time, and we're done. $\square$