## PNT STEP 4.

Goal: to show that

$$\frac{\psi_1(x)}{x^2} \sim \tfrac{1}{2} \text{ as } x \to \infty.$$

Recall: by STEP 3,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds,$$

for $c>1$ and $x \geq 1$. Note also that, by Lemma D of last time, for $x \geq 1$ and $c'>0$,

$$\frac{1}{2}\left(1-\tfrac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{x^s}{s(s+1)(s+2)}\, ds$$

$s \to s-1$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{(s-1)s(s+1)}\, ds,$$

where $c = c'+1 > 1$. So:

**Theorem 13.3.**
For $x \geq 1$ and $c>1$,

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1-\frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1} h(s)\, ds,$$

where $$h(s) = \frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\right).$$

Now parametrize the path of integration

by writing $s = c+it$, so that $ds = i\,dt$. Note that

$$x^{s-1} = x^{c+it-1} = x^{c-1}x^{it} = x^{c-1}e^{it\log x},$$

so we have

Corollary. For $x \geq 1$ and $c > 1$,

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{x^{c-1}}{2\pi}\int_{-\infty}^{\infty} h(c+it)e^{it\log x}\,dt. \quad (*)$$

We'll show that:

(A) $(*)$ remains true if $c = 1$,

(B) The right side of $(*)$, with $c = 1$, $\to 0$ as $x \to \infty$.

This will imply $\psi_1(x)/x^2 \to 1/2$ as $x \to \infty$, whence the PNT!

For (A), we need some facts about $\zeta(s)$.

Lemma E (roughly, Thm. 12.21 in Apostol).
(a) The series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

defines an analytic function of $s$ for $\mathrm{Re}(s) > 1$.
(b) $\zeta(s)$ has meromorphic continuation to $\mathrm{Re}(s) > 0$, its only pole there being a simple pole at $s = 1$, with residue 1. Moreover, for $\mathrm{Re}(s) > 0$ we have, for $N \in \mathbb{Z}_+$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - s\int_N^\infty \frac{x-[x]}{x^{s+1}}\,dx + \frac{N^{1-s}}{s-1},$$

and
$$\zeta'(s) = -\sum_{n=1}^{N} \frac{\log n}{n^s} + s\int_N^\infty \frac{(x-[x])\log x}{x^{s+1}}\,dx$$

$$-\int_N^\infty \frac{x-[x]}{x^{s+1}}\,dx - \frac{N^{1-s}\log N}{s-1} - \frac{N^{1-s}}{(s-1)^2}.$$

(c) For $\mathrm{Re}(s)>1$, $\zeta(s)$ has the "Euler product expansion"

$$\zeta(s) = \prod_p (1-p^{-s})^{-1}.$$

Proof.

(a) Since $|n^{-s}| = n^{-\mathrm{Re}(s)}$, the series defining $\zeta(s)$ converges for $\mathrm{Re}(s)>1$. One shows that it converges ucc on (meaning "uniformly on compact subsets of") the half-plane $\mathrm{Re}(s)>1$, so by Corollary $\mathbb{C}$-$\Sigma$, it's analytic there.

(b) Now for $\mathrm{Re}(s)>1$, apply ESF with $f(x)=x^{-s}$, $y=N\in\mathbb{Z}_+$, and $x=M\in\mathbb{Z}_+$ to get

$$\sum_{n=N+1}^{M} n^{-s} = \int_N^M t^{-s}\,dt - s\int_N^M (t-[t])t^{-s-1}\,dt$$
$$= \frac{M^{1-s}}{1-s} - \frac{N^{1-s}}{1-s} - s\int_N^M (t-[t])t^{-s-1}\,dt.$$

Let $M\to\infty$; since $\mathrm{Re}(s)>1$, we get

$$\sum_{n=N+1}^{\infty} n^{-s} = \frac{-N^{1-s}}{1-s} - s\int_N^{\infty} (t-[t])t^{-s-1}\,dt.$$

The left side is $\zeta(s) - \sum_{n=1}^{N} n^{-s}$, so the stated formula for $\zeta(s)$ follows, for $Re(s)>1$. But, since $t-[t]=O(1)$, the right side of this formula is, e.g. by Thm. I-3, seen to be meromorphic for $Re(s)>0$, its only pole there being a simple pole at $s=1$, with residue

$$\text{Res}_{s=1}\left(\frac{N^{1-s}}{s-1}\right) = \lim_{s\to 1} N^{1-s} = 1.$$

This yields the stated results for $\zeta(s)$.

The expression for $\zeta'(s)$ follows from differentiating the one for $\zeta(s)$.

(c) (Somewhat informal proof.)
By the geometric series formula, we have

$$(1-p^{-s})^{-1} = \sum_{m=0}^{\infty} p^{-ms}$$

for $p$ prime and $Re(s)>1$. So

$$\prod_p (1-p^{-s})^{-1} = \prod_p \left(\sum_{m=0}^{\infty} p^{-ms}\right).$$

Upon expanding out the right side, one summand will equal 1, while every other summand will be of the form

$$p_1^{-m_1 s} \, p_2^{-m_2 s} \cdots p_k^{-m_k s} = n^{-s},$$

where $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. Since each positive integer $n > 1$ will arise this way exactly once, we have

$$\prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s).$$

(See Thm. C-4 for details on convergence etc.) $\square$