Recall: to complete STEP 3, we want to show:

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds,$$

for $x, c > 1$.

Lemma D (= Apostol, Lemma 3, ch. 13).

For $k \in \mathbb{Z}_+$, $u>0$, and $c>0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)} = \begin{cases} (1-u)^k/k! & \text{if } 0<u\le 1, \\ 0 & \text{if } u>1, \end{cases}$$

the integral being absolutely convergent.

Proof.

Absolute convergence is because, for k, u, c as stated, and for $s = c+it$ with $c, t \in \mathbb{R}$,

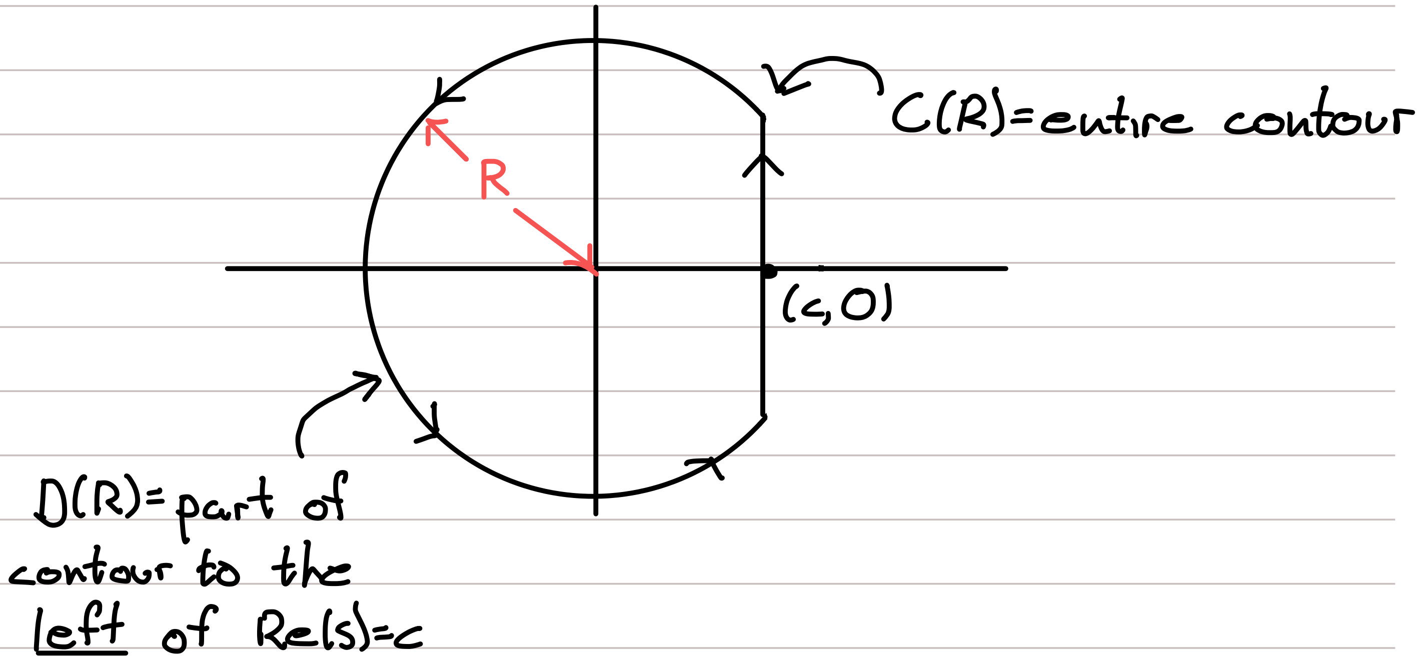
$$\left|\frac{u^{-s}}{s(s+1)\cdots(s+k)}\right| \le \frac{u^{-c}}{|s(s+1)|} \le \frac{u^{-c}}{\sqrt{(c^2+t^2)((c+1)^2+t^2)}}$$

$$\le \frac{u^{-c}}{c^2+t^2},$$

and our path of integration is $\{c+it : t \in (-\infty,\infty)\}$.

Now:

(A) Suppose $0<u\le 1$. For any $R > \max\{2k, c\}$, let $C(R)$ and $D(R)$ be the following contours in $\mathbb{C}$:



Note that all the poles of $u^{-s}/(s(s+1)\cdots(s+k))$ lie inside $C(R)$, since $R > 2k > k$. So, by the residue theorem,

$$\frac{1}{2\pi i}\int_{C(R)} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)}$$

$$= \sum_{n=0}^{k} \operatorname{Res}_{s=-n}\left(\frac{u^{-s}}{s(s+1)\cdots(s+k)}\right)$$

$$= \sum_{n=0}^{k} \frac{u^{n}}{(-n)(-n+1)\cdots(-1)(1)(2)\cdots(-n+k)}$$

$$= \sum_{n=0}^{k} \frac{u^{n}}{(-1)^{n} n!(k-n)!} = \frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}(-u)^{n}$$

$$= \frac{(1-u)^{k}}{k!}. \qquad (A1)$$

On the other hand, again by the residue theorem, and the fact that no poles of $u^{-s}/(s(s+1)\cdots(s+k))$ are outside $C(R)$,

$$\int_{C(R)} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)} = \lim_{R\to\infty} \int_{C(R)} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)}. \qquad (A2)$$

The last step is for the following reasons.

1) Since $0 < u \le 1$, $|u^{-s}| = u^{-\mathrm{Re}(s)}$ is bounded, on $D(R)$, by u^{-c};
2) On $D(R)$ we have, for $0 \le n \le k$,
$|s+n| \geq |s| - |n| = R - n \geq R - k > R/2$,

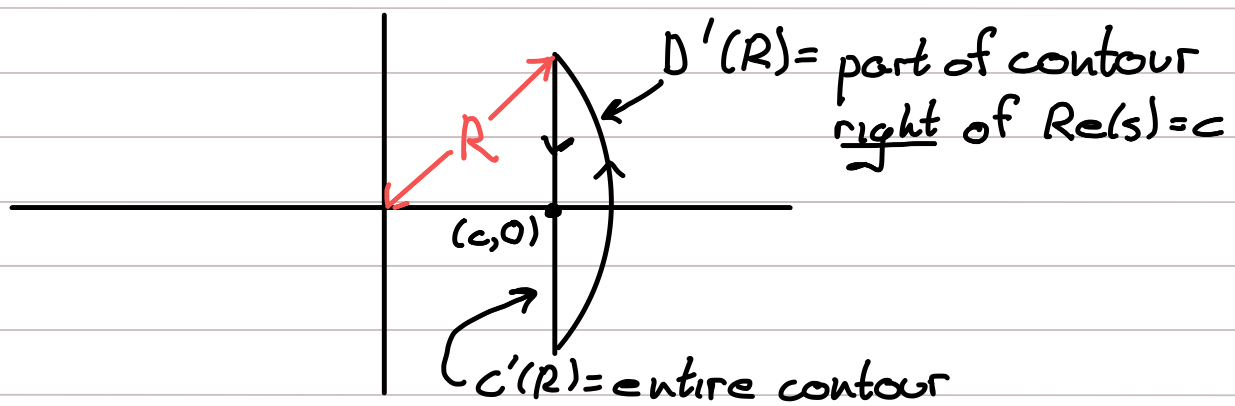
and consequently

$$\lim_{R\to\infty} \left| \int_{D(R)} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)} \right| < \lim_{R\to\infty} \frac{u^{-c}\cdot \mathrm{length}(D(R))}{(R/2)^{k+1}}$$

$$= \lim_{R\to\infty} O(R^{-k}) = 0.$$

Comparing (A1) and (A2) gives us our result for $0 < u \le 1$.

(B) Now suppose $u > 1$. For any $R > c$, consider contours $C'(R)$ and $D'(R)$ as follows:



Since $u^{-s}/(s(s+1)\cdots(s+k))$ has no poles inside $D'(R)$, we have, by Cauchy's Theorem,

$$0 = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{C'(R)} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)}$$

$$= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)} + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{D'(R)} \frac{u^{-s}\,ds}{s(s+1)\cdots(s+k)}.$$

Using arguments like the above, and noting that $|u^{-s}|$ decreases as $\mathrm{Re}(s)$ increases, we find that the limit on the right is 0, and we're done. $\square$

Now note the following: for $c > 1$,

Corollary of last time

$$\frac{\psi_1(x)}{x^2} = \frac{1}{x^2}\sum_{n\le x}\Lambda(n)(x-n) = \frac{1}{x}\sum_{n\le x}\Lambda(n)(1-n/x)$$

$$= \frac{1}{x}\sum_{n=1}^{\infty}\Lambda(n)\cdot\begin{cases}(1-n/x) & \text{if } n\le x,\\ 0 & \text{if } n>x\end{cases}$$

$$= \frac{1}{x}\sum_{n=1}^{\infty}\Lambda(n)\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(n/x)^{-s}}{s(s+1)}\,ds$$

Lemma D with $u = n/x$ and $k=1$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s(s+1)}\sum_{n=1}^{\infty}\Lambda(n)n^{-s}$$

the switch of sum and integral can be justified, for $c>1$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds,$$

Lemma C of last time

and STEP 3 is done!!