PNT, continued.

Recall: we've shown

STEP 1: $\psi(x) \sim x \Rightarrow \pi(x) \sim x/\log x$,
where
$$\psi(x) = \sum_{n \le x} \Lambda(n).$$

On to

STEP 2. Show that, if we define
$$\psi_1(x) = \int_1^x \psi(t)\,dt,$$
then $\psi_1(x) \sim x^2/2 \Rightarrow \psi(x) \sim x$.

This follows from a sort of "l'Hôpital's rule:"

Lemma A (Apostol, ch. 13, Lemma 2.)

Suppose $a(n) \geq 0$ for $n \in \mathbb{Z}_+$,
$A(x) = \sum_{n \le x} a(n)$, and $A_1(x) = \int_1^x A(t)\,dt$.

Then
$$A_1(x) \sim Lx^c \Rightarrow A(x) \sim cLx^{c-1},$$
for $c, L \in \mathbb{R}$.

Proof
Since $a(n) \geq 0$ $\forall n$, $A(x)$ is nondecreasing.

So, if $\beta > 1$, then

$$A_1(\beta x) - A_1(x) = \int_x^{\beta x} A(u)\,du \geq A(x)\int_x^{\beta x} du$$

$$= A(x)\cdot x(\beta - 1). \quad \text{So}$$

$$xA(x) \leq \frac{1}{\beta - 1}\left[A_1(\beta x) - A_1(x)\right]. \quad \text{So for } c \in \mathbb{R},$$

$$\frac{A(x)}{x^{c-1}} \leq \frac{1}{\beta - 1}\left[\frac{A_1(\beta x)}{(\beta x)^c}\beta^c - \frac{A_1(x)}{x^c}\right]$$

or, letting $x \to \infty$,

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \leq \frac{1}{\beta - 1}\left(L\beta^c - L\right) = \frac{L(\beta^c - 1)}{\beta - 1}.$$

This is true for all $\beta > 1$, so it's true in the limit as $\beta \to 1^+$. So

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \leq L \cdot \lim_{\beta \to 1+} \frac{\beta^c - 1}{\beta - 1} = Lc. \quad (*)$$

Similarly, if $0 < \alpha < 1$, then

$$A_1(x) - A_1(\alpha x) \leq A(x)\cdot x(1-\alpha),$$

so $\quad xA(x) \geq \dfrac{A_1(x) - A_1(\alpha x)}{1-\alpha},$ so for $c \in \mathbb{R}$,

$$\frac{A(x)}{x^{c-1}} \geq \frac{1}{1-\alpha}\left[\frac{A_1(x)}{x^c} - \frac{A_1(\alpha x)}{(\alpha x)^c}\alpha^c\right].$$

Letting $x \to \infty$ gives

$$\liminf_{x\to\infty} \frac{A(x)}{x^{c-1}} \geq \frac{1}{1-\alpha}\left(L - L\alpha^c\right) = \frac{L(1-\alpha^c)}{1-\alpha}.$$

This is true for all $0<\alpha<1$, so it's true as $\alpha \to 1^-$, so

$$\liminf_{x\to\infty} \frac{A(x)}{x^{c-1}} \geq L \lim_{\alpha\to 1^-} \frac{1-\alpha^c}{1-\alpha} = Lc. \qquad (**)$$

Together, $(*)$ and $(**)$ give the result. □

STEP 2 follows; on to STEP 3.

We wish to show that, for $x, c > 1$,

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds.$$

We begin with:

<u>Lemma B</u> (Apostol, Lemma 1, Ch. 13)

If $a(n)$ is an arithmetic function and

$$A(x) = \sum_{n\leq x} a(n),$$

then

$$\sum_{n\leq x} (x-n)a(n) = \int_1^x A(t)\,dt.$$

Proof.
By ASF with $f(x)=x$ and $y=\frac{1}{2}$,

$$\sum_{n\le x} na(n) = xA(x)-\tfrac{1}{2}A(\tfrac{1}{2}) - \int_{1/2}^{x} A(t)\,dt$$
$$= xA(x)-\int_1^x A(t)\,dt,$$

since $A(x)=0$ for $x<1$. Since

$$xA(x)=\sum_{n\le x} x\,a(n),$$

the result follows. □

Putting $a(n)=\Lambda(n)$ into Lemma 1 gives

Corollary $\psi_1(x)=\sum_{n\le x}(x-n)\Lambda(n)$.

For the next step, we'll need

Lemma C (from somewhere in Apostol).

For $\mathrm{Re}(s)>1$, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\Lambda(n)n^{-s}.$$

Proof. We've seen (HW #4, Exercise II(A)) that, for $\mathrm{Re}(s)>1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s}.$$

Also, by term-by-term differentiation,

$$\zeta'(s) = -\sum_{n=1}^{\infty} \log n \; n^{-s}, \qquad \text{for } \operatorname{Re} s > 1.$$

So, for such s,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{d=1}^{\infty} \sum_{c=1}^{\infty} \mu(d) \log c \, (cd)^{-s}$$

put $n=cd$ $$= \sum_{n=1}^{\infty} n^{-s} \sum_{d \mid n} \mu(d) \log(n/d)$$

$$= \sum_{n=1}^{\infty} \Lambda(n) n^{-s},$$

by Thm. 2.11. $\square$