The Prime Number Theorem (PNT).

It says
$$\pi(x) \sim \frac{x}{\log x} \quad \left(\text{that is, } \lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1\right)$$

where, for $x \in \mathbb{R}_+$,
$$\pi(x) = \sum_{n \leq x} 1.$$

Strategy of proof:

STEP 1.
Show PNT is implied by the statement

$$\psi(x) \sim x, \qquad (*)$$

where by definition
$$\psi(x) = \sum_{n \leq x} \Lambda(n),$$

Λ being Mangoldt's function.

STEP 2.
Show that $(*)$ is implied by the statement

$$\psi_1(x) \sim x^2/2, \qquad (**)$$

where
$$\psi_1(x) = \int_1^x \psi(t)\,dt.$$

STEP 3. Show that

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)}\left(\frac{-\zeta'(s)}{\zeta(s)}\right) ds \qquad (***)$$

for any $x, c > 1$.

STEP 4 (the big one).

Show the right side of (***) approaches $1/2$ as $x \to \infty$.

This will do it!! So on to

STEP 1 (Apostol sections 4.1-4.4).

Let
$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

where
$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of the prime } p, \\ 0 & \text{if not.} \end{cases}$$

We have:

Theorem 4.1.

Let
$$\theta(x) = \sum_{p \le x} \log p.$$

Then for $x \ge 1$,
$$0 \le \frac{\psi(x)}{x} - \frac{\theta(x)}{x} \le \frac{\log^2 x}{2 \log 2 \sqrt{x}}.$$

Proof.

We have
$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p : p^m \le x} \log p$$
$$= \sum_{m=1}^{\infty} \sum_{p \le x^{1/m}} \log p.$$

But note that the sum on p is empty if $m > \log_2 x = \log x / \log 2$, because in this case $1/m \log x < \log 2$, so

$$x^{1/m} = e^{\frac{1}{m}\log x} < e^{\log 2} = 2.$$

So
$$\psi(x) = \sum_{m \le \log_2 x} \sum_{p \le x^{1/m}} \log p = \sum_{m \le \log_2 x} \theta(x^{1/m}).$$

But then
$$\psi(x) - \theta(x) = \sum_{2 \le m \le \log_2 x} \theta(x^{1/m}).$$

Each summand on the right is ≥ 0, so $\psi(x) - \theta(x) \ge 0$. Moreover,
$$\theta(x) = \sum_{p \le x} \log p \le \sum_{p \le x} \log x \le x \log x,$$

so
$$0 \le \psi(x) - \theta(x) \le \sum_{2 \le m \le \log_2 x} x^{1/m} \log x^{1/m}$$

$$\le x^{1/2} \log x^{1/2} \sum_{1 \le m \le \log_2 x} 1$$

$$\le x^{1/2} \cdot \tfrac{1}{2} \log x \cdot \log_2 x$$

$$= \sqrt{x}\, \frac{\log^2 x}{2 \log 2}.$$

Divide by x to get the result. $\square$

Note Thm. 4.1 implies that

$$\lim_{x \to \infty} \frac{\psi(x)}{x} - \frac{\theta(x)}{x} = 0.$$

So $\psi(x) \sim x$ implies $\theta(x) \sim x$. If we can show that $\theta(x) \sim x$ implies $\pi(x) \sim x/\log x$, then we'll be done with STEP 1.

Theorem 4.4.
If $\theta(x) \sim x$, then $\pi(x) \sim x/\log x$.

Proof
Suppose $\theta(x) \sim x$: then $\lim_{x\to\infty} \theta(x)/x = 1$,

so $\theta(x) = x + R(x)$, where $\lim_{x\to\infty} R(x)/x = 0$.

Now apply ASF, with
$$A(x) = \theta(x) = \sum_{n \le x} a(n) \log n,$$

where $a(n) = 1$ for n prime and 0 otherwise, and with $f(x) = 1/\log x$, $y = 3/2$.

We get

$$\pi(x) = \sum_{p \le x} 1 = \sum_{n \le x} a(n) \log n\, f(n)$$

$$= \frac{\theta(x)}{\log x} - \frac{\theta(3/2)}{\log 3/2} + \int_{3/2}^{x} \frac{\theta(t)}{t \log^2 t}\, dt$$

$$= \frac{x + R(x)}{\log x} + \int_{2}^{x} \frac{t + R(t)}{t \log^2 t}\, dt$$

$$= \frac{x}{\log x} + \frac{R(x)}{\log x} + \int_{2}^{x} \frac{t + R(t)}{t \log^2 t}\, dt.$$

so

$$\frac{\pi(x)}{x/\log x} = 1 + \frac{R(x)}{x} + \frac{\log x}{x} \int_{2}^{x} \frac{t + R(t)}{t \log^2 t}\, dt.$$

We want to show that the left side approaches 1 as $x \to \infty$. Since $R(x)/x \to 0$ as $x \to \infty$, it's enough to show that

$$\frac{\log x}{x} \int_2^x \frac{t + R(t)}{t \log^2 t}\, dt \to 0 \quad \text{as} \quad x \to \infty,$$

which will certainly be the case if the integral is $O(x/\log^2 x)$. But, since $R(t)/t \to 0$ as $t \to 0$, we have $R(t) = O(t)$, so

$$\int_2^x \frac{t + R(t)}{t \log^2 t}\, dt = O\left(\int_2^x \frac{t}{t \log^2 t}\, dt\right) = O\left(\int_2^x dt/\log^2 t\right)$$

$$= O\left(\int_2^{\sqrt{x}} dt/\log^2 t + \int_{\sqrt{x}}^x dt/\log^2 t\right)$$

$$= O(\sqrt{x}/\log^2 2) + O((x - \sqrt{x})/\log^2 \sqrt{x})$$

$$= O(x/\log^2 x),$$ and we're done

(with STEP 1)! $\square$