More $L(1,\chi) \neq 0$.

Recall: to show $L(1,\chi)\neq 0$ for χ real and nonprincipal, we need only prove:

Theorem 6.20(b) (slight return).

Suppose χ is real and nonprincipal,

$$f(n) = \chi(n)/\sqrt{n}, \quad g(n) = 1/\sqrt{n},$$

$$F(x) = \sum_{n\le x} f(n), \qquad G(x) = \sum_{n \le x} g(n),$$

and

$$B(x) = \sum_{n\le\sqrt{x}} f(n)G(x/n) + \sum_{n\le\sqrt{x}} g(n)F(x/n) - F(\sqrt{x})G(\sqrt{x}).$$

Then $B(x) = 2\sqrt{x}\,L(1,\chi) + O(1)$.

Proof.
Recall that, for certain constants $A, B > 0$:

(i) $G(x) = \sum_{n\le x} \frac{1}{\sqrt{n}} = 2\sqrt{x} + A + O(1/\sqrt{x})$ (Thm. 3.2(b)),

(ii) $F(x) = \sum_{n\le x} \frac{\chi(n)}{\sqrt{n}} = B + O(1/\sqrt{x})$ (Thm. 6.18(c)),

(iii) $\sum_{n\le x} \frac{\chi(n)}{n} = L(1,\chi) + O(1/x)$ (Thm. 6.18(a)).

So

(i) and (ii)

$$B(x) = \sum_{n\le\sqrt{x}} \frac{\chi(n)}{\sqrt{n}}\left(2\sqrt{\frac{x}{n}} + A + O\left(\sqrt{\frac{n}{x}}\right)\right)$$

$$+ \sum_{n\le\sqrt{x}} \frac{1}{\sqrt{n}}\left(B + O\left(\sqrt{\frac{n}{x}}\right)\right)$$

$$- (B + O(x^{-1/4}))(2x^{1/4} + A + O(x^{-1/4}))$$

$$= 2\sqrt{x}\sum_{n\le\sqrt{x}} \frac{\chi(n)}{n} + A\sum_{n\le\sqrt{x}} \frac{\chi(n)}{\sqrt{n}} + O\left(\frac{1}{\sqrt{x}}\sum_{n\le\sqrt{x}} 1\right)$$

$$+ B\sum_{n\le\sqrt{x}} \frac{1}{\sqrt{n}} + O\left(\frac{1}{\sqrt{x}}\sum_{n\le\sqrt{x}} 1\right) - 2Bx^{1/4} + O(1)$$

(iii), (ii), (i)

$$= 2\sqrt{x}\,(L(1,\chi) + O(1/\sqrt{x})) + A(B + O(x^{-1/4}))$$

$$+ O(1) + B(\cancel{2x^{1/4}} + A + O(x^{-1/4})) + O(1) - \cancel{2Bx^{1/4}} + O(1)$$

$$= 2\sqrt{x}\, L(1,\chi) + O(1). \qquad \square$$

Finally, we need only show that $L(1,\chi) \neq 0$ for χ nonreal. To this end:

Lemma 7.8.
If χ is nonprincipal and $L(1,\chi) = 0$, then

$$L'(1,\chi)\sum_{n\le x} \frac{\mu(n)\chi(n)}{n} = \log x + O(1)$$

for $x \ge 1$.

<u>Proof.</u> Assume $L(1,\chi)=0$.

Recall:

(iv) $$\sum_{n\le x}\frac{\chi(n)\log n}{n}=-L'(1,\chi)+O\left(\frac{\log x}{x}\right)\quad \text{(Thm 6.18(b))},$$

(v) $$\sum_{n\le x}\log n = x\log x - x + O(\log x)\quad \text{(Thm. 3.15)}.$$

Now define $F(x)=x\log x$ for $x\ge 1$, and

$G(x)=\sum_{n\le x}\chi(n)F(x/n)$. Then

$$G(x)=\sum_{n\le x}\chi(n)\cdot\frac{x}{n}\log(x/n)$$

$$=x\log x\sum_{n\le x}\frac{\chi(n)}{n}-x\sum_{n\le x}\frac{\chi(n)}{n}\log n$$

(iii), (iv)
$$=x\log x\,(L(1,\chi)+O(1/x))+x\left(L'(1,\chi)+O\left(\frac{\log x}{x}\right)\right)$$

$$=xL'(1,\chi)+O(\log x),$$

since $L(1,\chi)$ by assumption. But then, by generalized Möbius inversion, since χ is completely multiplicative,

$$x\log x = F(x)=\sum_{n\le x}\mu(n)\chi(n)G(x/n)$$

$$=\sum_{n\le x}\mu(n)\chi(n)\left(\frac{x}{n}L'(1,\chi)+O\left(\log\left(\frac{x}{n}\right)\right)\right)$$

$$=xL'(1,\chi)\sum_{n\le x}\frac{\mu(n)\chi(n)}{n}+O\left(\sum_{n\le x}\log(x/n)\right)$$

or, dividing by x,

$$\log x = L'(1,\chi)\sum_{n\le x}\frac{\mu(n)\chi(n)}{n} + O\left(\frac{1}{x}\sum_{n\le x}\log(x/n)\right),$$

and we need only show the quantity in $O(\;\;)$ is $O(1)$.

But

$$\frac{1}{x}\sum_{n\le x}\log(x/n) = \frac{\log x}{x}\sum_{n\le x}1 - \frac{1}{x}\sum_{n\le x}\log n$$

$$\overset{(v)}{=} \frac{\log x}{x}(x+O(1)) - \frac{1}{x}(x\log x - x + O(\log x))$$

$$\log x + O\left(\frac{\log x}{x}\right) - \log x + 1 + O\left(\frac{\log x}{x}\right)$$

$$= O(1). \qquad \square$$

Next, let

$$N(k) = |\{\text{nonreal } \chi \bmod k : L(1,\chi)=0\}|.$$

To prove Dirichlet's theorem, it now suffices to show $N(k)=0$.

To this end, note that $N(k)$ is <u>even</u>, since $L(1,\chi)=0 \iff 0=\overline{L(1,\chi)}=L(1,\overline{\chi})$, and for χ nonreal, $\chi\ne\overline{\chi}$.

<u>Lemma 7.7.</u>

$$\sum_{\substack{p\le x\\ p\equiv 1 (\mathrm{mod}\ k)}}\frac{\log p}{p} = \frac{1-N(k)}{\varphi(k)}\log x + O(1).$$

[This proves $N(k)=0$: if not then $N(k)\geq 2$, meaning the right side above $\to -\infty$ as $x\to\infty$. But the left side is ≥ 0.]

Proof.

By Lemmas 7.4 and 7.5,

$$\sum_{\substack{p\leq x \\ p\equiv 1 \,(\mathrm{mod}\, k)}} \frac{\log p}{p} = \frac{\log x}{\varphi(k)} + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \sum_{p\leq x} \frac{\chi_r(p)\log p}{p} + O(1)$$

$$= \frac{\log x}{\varphi(k)} + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \left[-L'(1,\chi_r) \sum_{n\leq x} \frac{\mu(n)\chi(n)}{n} \right] + O(1).$$

Now if $L(1,\chi_r)=0$, the summand in [ ] is $-\log x + O(1)$, by Lemma 7.8. If not (in particular, if χ_r is real), it's $O(1)$ by Lemma 7.6. So

$$\sum_{\substack{p\leq x \\ p\equiv 1 \,(\mathrm{mod}\, k)}} \frac{\log p}{p} = \frac{\log x}{\varphi(k)} + \frac{1}{\varphi(k)} \cdot N(k)(-\log x) + O(1)$$

$$= \frac{1-N(k)}{\varphi(k)} \log x + O(1).$$

$\square$

DONE!!