# Dirichlet's Theorem, continued.

Recall:

Dirichlet's Theorem, which says:
if $h>0$ and $(h,k)=1$,
then $|\{\text{primes } p : p \equiv h \pmod k\}| = \infty$,

is implied by

**Thm. 7.3.** For $(h,k)=1$ and $x \geq 1$,

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod k}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O(1).$$

Towards Thm. 7.3, we've proved:

**Lemma 7.4.** For $(h,k)=1$ and $x \geq 1$,

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod k}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \overline{\chi_r(h)} \sum_{p \leq x} \frac{\chi_r(p) \log p}{p} + O(1).$$

So we need only show that, for $2 \leq r \leq \varphi(k)$, the sum on $p$, on the right, is $O(1)$.

To this end, we'll show:

**Lemma 7.5.**
For $x \geq 1$ and $\chi \neq \chi_1$,

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = -L'(1,\chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} + O(1),$$

where
$$L'(1,\chi) = -\sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n}$$
(which is finite by Thm 6.17 or Thm. 6.18(b)).

[Dirichlet's thm. will then follow if we can show that
$$\sum_{n\le x} \frac{\mu(n)\chi(n)}{n} = O(1).]$$

Proof of Lemma 7.5.

Let
$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of a prime } p, \\ 0 & \text{if not.} \end{cases}$$

Then
$$\sum_{n\le x} \frac{\chi(n)\Lambda(n)}{n} = \sum_{a=1}^{\infty} \sum_{p^a\le x} \frac{\chi(p^a)\Lambda(p^a)}{p^a}$$
$$= \sum_{p\le x} \frac{\chi(p)\log p}{p} + \sum_{a=2}^{\infty} \sum_{p^a\le x} \frac{\chi(p^a)\log p}{p^a}.$$

Now the double sum on the right is bounded in absolute value by
$$\sum_{a=2}^{\infty} \sum_{n=2}^{\infty} \frac{\log n}{n^a} = \sum_{n=2}^{\infty} \log n \sum_{a=2}^{\infty} n^{-a} = \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)}$$

$< \infty$ (independently of $x$). So
$$\sum_{p\le x} \frac{\chi(p)\log p}{p} = \sum_{n\le x} \frac{\chi(n)\Lambda(n)}{n} + O(1).$$

But by Thm. 2.11,

$$\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d),$$

so

$$\sum_{p \le x} \frac{\chi(p)\log p}{p} = \sum_{n \le x} \frac{\chi(n)}{n} \sum_{d|n} \mu(d) \log(n/d) + O(1)$$

write $n = cd$

$$= \sum_{d \le x} \mu(d) \sum_{c \le x/d} \frac{\chi(cd)\log(c)}{cd} + O(1)$$

$$= \sum_{d \le x} \frac{\mu(d)\chi(d)}{d} \sum_{c \le x/d} \frac{\chi(c)\log c}{c} + O(1)$$

by Thm. 6.18(b), since $d \le x \Rightarrow \frac{x}{d} \ge 1$

$$= \sum_{d \le x} \frac{\mu(d)\chi(d)}{d}\left(-L'(1,\chi) + O\left(\frac{\log(x/d)}{x/d}\right)\right) + O(1)$$

$$= -L'(1,\chi)\sum_{d \le x} \frac{\mu(d)\chi(d)}{d} + O\left(\frac{1}{x}\sum_{d \le x} \log(x/d)\right) + O(1).$$

We'll be done if we can show that the quantity inside the first $O(\ )$ on the right is $O(1)$. But

$$\frac{1}{x}\sum_{d \le x} \log(x/d) = \frac{1}{x}\left(\sum_{d \le x} \log x - \sum_{d \le x} \log d\right)$$

Thms. 3.12(b) and 3.15

$$= \frac{1}{x}\left([x]\log x - (x\log x - x + O(\log x))\right)$$

$$= \frac{1}{x}\left((x + O(1))\log x - x\log x + x + O(\log x)\right)$$

$$= \frac{1}{x}\left(O(\log x) + x + O(\log x)\right) = O(1),$$

and we're done.

$\square$

So, again, we need only show that

$$\sum_{n\le x} \mu(n)\chi(n)/n = O(1). \qquad (\chi\ne\chi_1,\ x\ge 1).$$

We'll prove:

Lemma 7.6.
For $\chi\ne\chi_1$ and $x\ge 1$,

$$L(1,\chi)\sum_{n\le x} \mu(n)\chi(n)/n = O(1).$$

[Then we'll still need to show that $L(1,\chi)\ne 0$, for $\chi\ne\chi_1$.]

Proof of Lemma 7.6.
Recall generalized Möbius inversion (Thm. 2.23): If $F, G : \mathbb{R}^+ \to \mathbb{C}$ satisfy $F(x)=G(x)=0$ for $x\in(0,1)$, and $\alpha$ is an arithmetic function, then

$$G(x)=\sum_{n\le x}\alpha(n)F(x/n)$$

$$\iff F(x)=\sum_{n\le x}\mu(n)\alpha(n)G(x/n).$$

Apply this with $\alpha=\chi$ and $F(x)=x$ $(x\ge 1)$. Then

$$G(x)=\sum_{n\le x}\chi(n)\frac{x}{n} = x\sum_{n\le x}\frac{\chi(n)}{n},$$

so for $x\ge 1$,

$$x = F(x) = \sum_{n\le x}\mu(n)\chi(n)G\left(\frac{x}{n}\right)$$

$$= x \sum_{n \le x} \frac{\mu(n)\chi(n)}{n} \sum_{m \le x/n} \frac{\chi(m)}{m}.$$

By Thm. 6.18(a), the sum on the far right is $L(1,\chi) + O(n/x)$, for $x/n \ge 1$ (i.e. $n \le x$). So

$$x = x \sum_{n \le x} \frac{\mu(n)\chi(n)}{n}\left(L(1,\chi) + O(n/x)\right)$$

$$= x L(1,\chi) \sum_{n \le x} \frac{\mu(n)\chi(n)}{n} + x \cdot O\left(\sum_{n \le x} \frac{1}{x}\right)$$

$$= x L(1,\chi) \sum_{n \le x} \frac{\mu(n)\chi(n)}{n} + O(x).$$

Divide by $x$ to get the result. □

So now, we need only show $L(1,\chi) \ne 0$, for $\chi \ne \chi_1$. We will need to consider two separate cases:

(a) $\chi$ is real (i.e. real-valued),
(b) $\chi$ is nonreal (i.e. not real).