Dirichlet characters, continued.

Throughout, $k \in \mathbb{Z}^+$ is fixed, χ is a Dirichlet character mod k, and χ_1 is the principal one.

Recall Thm. 6.17: if $\chi \neq \chi_1$, f is decreasing, nonnegative, and continuously differentiable on $[x_0, \infty)$ for some $x_0 \in \mathbb{R}$, and $f(x) \to 0$ as $x \to \infty$, then for $x \geq x_0$,

$$\sum_{n \leq x} \chi(n) f(n) = \sum_{n=1}^{\infty} \chi(n) f(n) + O(f(x)),$$

the sum on the right being convergent.

Applying this with $f(x) = 1/x$, $\frac{\log x}{x}$, and $\sqrt{x}$ respectively gives

Thm. 6.18 If $\chi \neq \chi_1$ and $x \geq 1$, then

(a) $$\sum_{n \leq x} \frac{\chi(n)}{n} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} + O(1/x);$$

(b) $$\sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n} + O\left(\frac{\log x}{x}\right);$$

(c) $$\sum_{n \leq x} \frac{\chi(n)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}} + O(1/\sqrt{x}).$$

Remark: for $\chi \neq \chi_1$, define the "Dirichlet series"

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}.$$

By Thm. 6.17, $L(s,\chi)$ converges for $s>0$. In fact, one shows it converges uniformly for s in compact subsets of $(0,\infty)$. So we can differentiate term by term, for $s>0$:

$$L'(s,\chi)=-\sum_{n=1}^{\infty}\chi(n)\log n\, n^{-s}.$$

So Thm. 6.18 reads:

Thm 6.18, reprise.

For $\chi\neq\chi_1$ and $x\geq 1$,

(a) $\displaystyle\sum_{n\leq x}\frac{\chi(n)}{n}=L(1,\chi)+O(1/x)$,

(b) $\displaystyle\sum_{n\leq x}\frac{\chi(n)\log n}{n}=-L'(1,\chi)+O\left(\frac{\log x}{x}\right)$;

(c) $\displaystyle\sum_{n\leq x}\frac{\chi(n)}{\sqrt{n}}=L(1/2,\chi)+O(1/\sqrt{x})$.

Dirichlet's theorem on primes in arithmetic progression.

Theorem:

If $h>0$ and $(h,k)=1$, then the progression

$h, h+k, h+2k, \ldots$

contains infinitely many primes.

Note that Dirichlet's theorem is implied by:

Theorem 7.3

If $h>0$ and $(h,k)=1$, then $\forall x \geq 1$,

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O(1).$$

[Remark:

recall Thm. 4.10:

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

This and Thm. 7.3 imply that primes are, in a sense, evenly distributed among the $\varphi(k)$ equivalence classes $h \bmod k$ such that $(h,k)=1$.]

To prove Thm. 7.3, we'll need some lemmas.

Lemma 7.4 For $x \geq 1$ and $(h,k)=1$,

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \overline{\chi_r(h)} \sum_{p \leq x} \frac{\chi_r(p) \log p}{p} + O(1).$$

Proof.

We first rewrite $\log x$, as follows. By Thm. 4.10 and by adding and subtracting,

$$\log x = \sum_{p \le x} \frac{\log p}{p} + O(1)$$

$$= \sum_{p \le x} \overline{\chi_1(h)}\chi_1(p) \frac{\log p}{p} + \sum_{p \le x} \left(1 - \overline{\chi_1(h)}\chi_1(p)\right) \frac{\log p}{p} + O(1).$$

Now note that $\overline{\chi_1(h)} = 1$, since $(h,k)=1$. So $1 - \overline{\chi_1(h)}\chi_1(p) = 1 - \chi_1(p)$. This is non-zero only if $\chi_1(p) \ne 1$, meaning $(p,k)>1$, meaning $p|k$. The number of $p \le x$ with $p|k$ is finite and independent of x, so $1 - \overline{\chi_1(h)}\chi_1(p)$ is nonzero only finitely often. So the second sum on the right side of the above equation for $\log x$ is $O(1)$. So

$$\log x = \sum_{p \le x} \overline{\chi_1(h)}\chi_1(p) \frac{\log p}{p} + O(1).$$

But then the right side of Lemma 7.4 equals

$$\frac{1}{\varphi(k)} \sum_{p \le x} \overline{\chi_1(h)}\chi_1(p) \frac{\log p}{p}$$

$$+ \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \overline{\chi_r(h)} \sum_{p \le x} \frac{\chi_r(p)\log p}{p} + O(1)$$

combine terms

$$= \frac{1}{\varphi(k)} \sum_{r=1}^{\varphi(k)} \overline{\chi_r(h)} \sum_{p \le x} \frac{\chi_r(p)\log p}{p} + O(1)$$

switch order of summation

$$= \frac{1}{\varphi(k)} \sum_{p \le x} \frac{\log p}{p} \sum_{r=1}^{\varphi(k)} \chi_r(p)\overline{\chi_r(h)} + O(1)$$

Orthogonality (Thm. 6.16)

$$= \sum_{\substack{p \le x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} + O(1),$$

and we're done. $\square$

Note: if we can show that the sum

$$\sum_{p \le x} \frac{\chi_r(p)\log p}{p}$$

appearing in Lemma 7.4 is $O(1)$ for each $r > 1$, we will have Thm. 7.3, and hence Dirichlet's theorem.