More characters.

(A) Orthogonality revisited.

Recall the orthogonality relations for a finite abelian group $G = \{a_1, a_2, \ldots, a_n\}$ with character group $\hat{G} = \{f_1, f_2, \ldots, f_n\}$, where f_1 is the principal character:

$$\frac{1}{n} \sum_{k=1}^{n} f_i(a_k) \overline{f_j(a_k)} = \delta_{ij} \qquad (OR_1)$$

$(1 \le i, j \le n)$ and

$$\frac{1}{n} \sum_{k=1}^{n} f_k(a_i) \overline{f_k(a_j)} = \delta_{ij} \qquad (OR_2)$$

$(1 \le i, j \le n)$.

($\delta_{ij} = 1$ if $i = j$, 0 if not).

Again, (OR_1) says $\{n^{-1/2} f_i : 1 \le i \le n\}$ is an orthonormal basis for the space of functions on G.

What about (OR_2)? Well, note that any $a_i \in G$ defines a character on $\hat{G}$ (i.e. an element of $\hat{\hat{G}}$) by

$$a_i(f_k) = f_k(a_i)$$

$(a_i \in G,\ f_k \in \hat{G})$.

So (OR_2) says: $\{n^{-1/2} a_i : 1 \le i \le n\}$, considered as characters on $\hat{G}$, forms an orthonormal basis for the space of functions on $\hat{G}$.

(B) Dirichlet characters

Def'n: a Dirichlet character χ mod k is an extension to $\mathbb{Z}$ of a character f on $(\mathbb{Z}/k\mathbb{Z})^*$, where the extension is defined by

$$\chi(n) = \chi_f(n) = \begin{cases} f(\bar{n}) & \text{if } (n,k)=1, \\ 0 & \text{if not.} \end{cases}$$

Note that:

(a) A Dirichlet character mod k is periodic, with period k.

Proof: Suppose $m, n \in \mathbb{Z}$ satisfy $m = n + \ell k$ for some $\ell \in \mathbb{Z}$. Then $(m,k) = (n+\ell k, k) = (n,k)$. If $(m,k)=(n,k)=1$, then $\chi(m) = f(\bar{m}) = f(\bar{n}) = \chi(n)$, while if $(m,k)=(n,k)>1$, then $\chi(m) = 0 = \chi(n)$.

(b) There are $\varphi(k)$ distinct Dirichlet characters mod k.

Proof: $|(\mathbb{Z}/k\mathbb{Z})^*| = \varphi(k)$, so $|\widehat{(\mathbb{Z}/k\mathbb{Z})^*}| = \varphi(k)$. Moreover, extensions χ_f and χ_g of distinct characters f and g on $(\mathbb{Z}/k\mathbb{Z})^*$ must be distinct, since $\chi_f(n) = \chi_g(n)\ \forall n$ $\Rightarrow f(\bar{n}) = g(\bar{n})\ \forall n \Rightarrow f = g$.

(c) Dirichlet characters are completely multiplicative.

Proof: if $(m,k)=(n,k)=1$, then $\chi(mn) = f(\overline{mn}) = f(\bar{m})f(\bar{n}) = \chi(m)\chi(n)$.
If $(m,k)>1$ or $(n,k)>1$, then either $\chi(m)=0$ or $\chi(n)=0$, so $\chi(m)\chi(n)=0$. Moreover, in this case, $(mn,k)>1$, so $\chi(mn) = 0 = \chi(m)\chi(n)$.

(d) Any completely multiplicative function χ on $\mathbb{Z}$ with period k, and such that $\chi(n)=0$ for $(n,k)>1$, is a Dirichlet character mod k.

Proof: for such a χ, define a function f on $(\mathbb{Z}/n\mathbb{Z})^*$ by $f(\bar{n})=\chi(n)$. One checks that f is a character, and that $\chi=\chi_f$.

Now write $\{\chi_1, \chi_2, \dots, \chi_{\varphi(k)}\}$ for the set of Dirichlet characters mod k, with χ_1 the principal Dirichlet character (i.e. $\chi_1 = \chi_{f_1}$). Then (OR_2), with the fact that $\chi_r(m)=0$ for $(m,k)>1$ and $1 \le r \le \varphi(k)$, yield

Theorem 6.16.

If $(n,k)>1$, then

$$\sum_{r=1}^{\varphi(k)} \chi_r(m)\overline{\chi_r(n)} = \begin{cases} \varphi(k) & \text{if } m \equiv n \pmod{k}, \\ 0 & \text{if not.} \end{cases}$$

(c) Sums of Dirichlet characters.

Theorem 6.17

Let χ be a nonprincipal Dirichlet character mod k. Let f be nonnegative, continuously differentiable, and decreasing on $[x_0, \infty)$ for some $x_0 \in \mathbb{R}$. Then for $x_0 \le x \le y$, we have

(a) $$\sum_{x<n\le y} \chi(n) f(n) = O(f(x)) \quad \text{for } x \ge x_0,$$

independently of y.

Moreover, if $\lim_{x\to\infty} f(x)=0$, then

(b) $\sum_{n=1}^{\infty} \chi(n) f(n)$ converges, and for $x \geq x_0$,

(c) $\sum_{n \leq x} \chi(n) f(n) = \sum_{n=1}^{\infty} \chi(n) f(n) + O(f(x))$.

Proof

Let $A(x) = \sum_{n \leq x} \chi(n)$. Then $A(x) = O(1)$ by

Apostol Ch. 6, Exercise 15 (see HW5.)

So by ASF, we have

$$\sum_{x < n \leq y} \chi(n) f(n) = f(y) A(y) - f(x) A(x) - \int_x^y A(t) f'(t)\,dt$$
$$= O(f(y)) + O(f(x)) + O\left(\int_x^y - f'(t)\,dt \right)$$
$$= O(f(y)) + O(f(x)) + O(f(x) - f(y)) = O(f(x)),$$

because f is decreasing, so $f(y) \leq f(x)$. This proves (a).

If $f(x) \to 0$ as $x \to \infty$, then (a) proves that the partial sums
$$\sum_{n \leq N} \chi(n) f(n)$$
form a Cauchy sequence, which proves (b).

Finally, if $\lim_{n \to \infty} f(x) = 0$, then

$$\sum_{n \leq x} \chi(n) f(n) = \sum_{n=1}^{\infty} \chi(n) f(n) - \sum_{n > x} \chi(n) f(n),$$

with both series on the right converging by part (b). Moreover,

$$\sum_{n>x} \chi(n)f(n) = \lim_{y\to\infty} \sum_{x<n\le y} \chi(n)f(n),$$

which is $O(f(x))$ by part (a). So part (c) is proved. $\square$

Applying Thm 6.26 with $f(x) = 1/x$, $\frac{\log x}{x}$, and $\sqrt{x}$ respectively gives

Thm. 6.18 If χ is nonprincipal mod k and $x \geq 1$, then

(a) $$\sum_{n\le x} \frac{\chi(n)}{n} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} + O(1/x);$$

(b) $$\sum_{n\le x} \frac{\chi(n)\log n}{n} = \sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n} + O\left(\frac{\log x}{x}\right);$$

(c) $$\sum_{n\le x} \frac{\chi(n)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} + O(1/\sqrt{x}).$$