More on characters.

(A) The character group $\hat{G}$.

Again, G is a finite abelian group with identity $\{e\}$. Write $|G| = n$.

Note that
$$\hat{G} = \{\text{characters on } G\}$$
is an abelian group under pointwise multiplication (i.e. $fg(a) = f(a)g(a)$ for $f, g \in \hat{G}$, $a \in G$).

The identity $f_1 \in \hat{G}$ is the principal character f_1 defined by $f_1(a) = 1 \ \forall a \in G$. The inverse f^{-1} of $f \in \hat{G}$ is given, for $a \in G$, by $f^{-1}(a) = 1/f(a) = \overline{f(a)}$, since $f(a)$ is a root of unity.

By Thm. 6.8 of last time, $|\hat{G}| = n$.

(B) Orthogonality of characters.

Write $G = \{a_1, a_2, \ldots, a_n\}$,
$\hat{G} = \{f_1, f_2, \ldots, f_n\}$.

We always assume that f_1 is the principal character.

Let A denote the matrix with $f_i(a_j)$ in the i^{th} row and j^{th} column ($1 \le i, j \le n$).

We have

Thm. 6.10. For $1 \le i \le n$, the sum

$$\sum_{k=1}^{n} f_i(a_k)$$

of the i^{th} row entries of A equals n if $i=1$ and 0 otherwise.

Proof

In the case $i=1$, we're simply adding 1 to itself n times.

If $i \neq 1$, then $\exists\ 1 \le l \le n$: $f_i(a_l) \neq 0$. Note that the set $\{a_l a_1, a_l a_2, \dots a_l a_n\}$ is just G again (since $a_l a_r = a_l a_s \Rightarrow a_r = a_s$). So, since f_i is a homomorphism,

$$\sum_{k=1}^{n} f_i(a_k) = \sum_{k=1}^{n} f_i(a_l a_k) = f_i(a_l) \sum_{k=1}^{n} f_i(a_k).$$

So

$$(1 - f_i(a_l)) \sum_{k=1}^{n} f_i(a_k) = 0;$$

since $f_i(a_l) \neq 1$, the sum must be zero. $\square$

Remark 1.

Let $f_i, f_j \in \hat{G}$. Note that $f_i \overline{f_j} = f_1$ iff

$$f_i = \left(\overline{f_j}\right)^{-1} = f_j.$$

So Thm 6.10 is often written

$$\sum_{k=1}^{n} f_i(a_k)\overline{f_j(a_k)} = \begin{cases} n & \text{if } f_i = f_j, \\ 0 & \text{if not} \end{cases} \qquad (OR_1)$$

$(1 \le i, j \le n)$.

Next, we show that A is invertible:

<u>Thm 6.11</u>. We have $A^{-1} = n^{-1}A^*$, where A^* is the conjugate transpose of A. That is,

$$AA^* = nI,$$

where I is the $n \times n$ identity matrix.

<u>Proof.</u>

Recall that A has $f_i(a_k)$ in its i^{th} row and k^{th} column. So, by defintion of A^*, A^* has $\overline{f_j(a_k)}$ in its k^{th} row and j^{th} column. So by defintion of matrix multiplication, A^*A has

$$\sum_{k=1}^{n} f_i(a_k)\overline{f_j(a_k)},$$

in the i^{th} row and j^{th} column, and the result follows from (OR_1). $\square$

Consequently,

<u>Thm. 6.12</u>. We have

$$\sum_{k=1}^{n} f_k(a_i)\overline{f_k(a_j)} = n \cdot \delta_{ij} \qquad (OR_2)$$

for $1 \le j \le n$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if not.} \end{cases}$

<u>Proof.</u>

Multiply the equation $AA^* = nI$ by A^{-1} on the left and A on the right, to get

$$A^*A = A^{-1}nIA = A^{-1}AnI = nI.$$

So A^*A has $n\delta_{ij}$ in its i^{th} row and j^{th} column. But by definition of matrix multiplication, it also has

$$\sum_{k=1}^{n} \overline{f_k(a_i)} f_k(a_j)$$

there. Now swap i and j to get the desired result. $\square$

Note that

$$\sum_{k=1}^{n} f_k(a_i)\overline{f_k(a_j)} = \sum_{k=1}^{n} f_k(a_i a_j^{-1}).$$

So putting $a_j = e$ into the above theorem gives

Theorem 6.13.

$$\sum_{k=1}^{n} f_k(a_i) = \begin{cases} n & \text{if } a_i = e, \\ 0 & \text{if not.} \end{cases}$$

Remark 2.

(OR_1) is called an orthogonality relation for the following reason. Consider the inner product

$$\langle f, g \rangle = \sum_{k=1}^{n} f(a_k)\overline{g(a_k)}$$

of functions f, g on G.

(OR_1) says $\langle f_i, f_j \rangle = n\delta_{ij}$ for $1 \le i, j \le n$, meaning $\{f_1, f_2, \dots, f_n\}$ is an orthogonal set.

In fact it's an orthogonal basis (whose elements have norm $\sqrt{n}$), since the space of functions on G has dimension $|G| = n$.