Characters, continued.

Throughout, G is a finite abelian group with identity e.

Define
$$\hat{G} = \{\text{characters on } G\}$$
$$= \{\text{homomorphisms } f: G \to \mathbb{C}^*\}.$$

Our goal for today is to show that $|\hat{G}| = |G|$.

Lemma.
Let G' be a proper subgroup of G; let $a \in G \setminus G'$ have indicator h in G' (i.e. $h = \min\{m \in \mathbb{Z}_+ : a^m \in G'\}$). Let

$$G'' = \{xa^l : x \in G', 0 \le l < h\}$$

(G'' is a group by Thm. 6.6). Then

$$|\widehat{G''}| = |\widehat{G'}| \cdot h.$$

Proof. Let G, G', a, h, G'' be as stated. It will suffice to prove the following:

(a) Each $f \in \widehat{G'}$ extends in h distinct ways to a character on G''.

(b) If $f, g \in \widehat{G'}$ and $f \neq g$, then no extension of f equals an extension of g.

(c) Any $F \in \widehat{G''}$ equals one of the extensions of part (a).

Proof of (b): this is clear because, if an extension $\tilde{f}$ of f equals an extension $\tilde{g}$ of g, then the restrictions of $\tilde{f}$ and $\tilde{g}$ to G' agree, so $f = g$.

Proof of (a). Let $f \in \widehat{G'}$.
Since $f(a^h) \in \mathbb{C}^*$, we know that $f(a^h)$ has h distinct h^{th} roots $R_0, R_1, \ldots, R_{h-1}$.
Then
$$R_j^h = f(a^h) \qquad (0 \le j < h). \qquad (*)$$

For each $0 \le j < h$, define a function f_j on G'' by
$$f_j(xa^\ell) = f(x) R_j^\ell$$
$$(x \in G', \ 0 \le \ell < h).$$

To prove part (a), it will suffice to show that
(i) $f_j \ne f_k$ for $0 \le j, k < h$.
(ii) f_j is a homomorphism from G'' to $\mathbb{C}^*$, for $0 \le j < h$.

Proof of (i): If $j \ne k$, then
$$f_j(a) = R_j \ne R_k = f_k(a).$$

Proof of (ii): Let $x, y \in G'$ and $0 \le \ell, m < h$. Write $\ell + m = hq + r$ with $0 \le r < h$. Then, for $0 \le j < h$,

$$f_j(xa^{l} \cdot ya^{m}) = f_j(xya^{l+m}) = f_j(xya^{hq+r})$$

$$= f_j(xy(a^h)^q a^r) = f(x)f(y)f(a^h)^q R_j^r$$

by (*)
$$= f(x)f(y)(R_j^{h})^q R_j^r = f(x)f(y)R_j^{hq+r}$$

$$= f(x)f(y)R_j^{l+m} = (f(x)R_j^{l})(f(x)R_j^{m})$$

$$= f_j(xa^l)f_j(ya^m).$$

So f_j is a homomorphism from G'' to $\mathbb{C}^*$.
So (a) is proved.

Proof of (c): If $F \in \widehat{G''}$, then for $x \in G'$ and $0 \le l < h$,

$$F(xa^l) = F(x)F(a)^l = F(x)R_j^l,$$ where $R_j = F(a)$ is an h^{th} root of $F(a^h)$ (because $R_j^h = F(a)^h = F(a^h)$). So $F = f_j$ for some j, where f is the restriction of F to G'. (Clearly $f \in \widehat{G'}$.) So (c) is proved.

$\square$

Next, we have:

<u>Theorem 6.8.</u> $|\hat{G}| = |G|$.

<u>Proof</u> For G' a proper subgroup of G and $a \in G \setminus G'$ of indicator h in G', write $\langle G'; a \rangle$ for the group $G'' = \{xa^l : x \in G', 0 \le l < h\}$ described above.

Define a nested sequence of subgroups

$$G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots \subseteq G_t = G$$

of G recursively, as follows.

(1) $G_1 = \{e\}$.
(2) If $G_1 = G$ then $t = 1$. If not then:
(3) Let $a_1 \in G \setminus G_1$; let $G_2 = \langle G_1; a_1 \rangle$.
If $G_2 = G$, then $t = 2$. If not then:
(4) Let $a_2 \in G \setminus G_2$; let $G_3 = \langle G_2, a_2 \rangle$.

And so on. Each G_r has at least one more element than G_{r-1}, so eventually $G_t = G$.

Write h_r for the indicator of a_r in G_r ($1 \le r \le t-1$). Then, by Thm. 6.6 of last time,

$$\begin{aligned}|G| = |G_t| = |G_{t-1}| h_{t-1} &= |G_{t-2}| h_{t-2} h_{t-1} \\ &= \dots = |G_1| h_1 h_2 \cdots h_{t-1} \\ &= h_1 h_2 \cdots h_{t-1}.\end{aligned}$$

On the other hand, by the above Lemma,

$$\begin{aligned}|\hat{G}| = |\widehat{G_t}| = |\widehat{G_{t-1}}| h_{t-1} &= |\widehat{G_{t-2}}| h_{t-2} h_{t-1} \\ &= \dots = |\widehat{G_1}| h_1 h_2 \cdots h_{t-1}.\end{aligned}$$

But $|\widehat{G_1}| = 1$ because the only nonzero homomorphism on $\{e\}$ is given by $f(e) = 1$. So

$$|\hat{G}| = h_1 h_2 \cdots h_{t-1} = |G|, \text{ and we're done.}$$

$\square$