The Abel summation formula (ASF)

(a generalization of ESF).

Thm. 4.2.

Given an arithmetic function a, let

$$A(x) = \sum_{n \le x} a(n) \qquad (x \ge 0).$$

Assume $0 < y < x$, and f is continuously differentiable on $[y, x]$. Then

$$\sum_{y < n \le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\, dt.$$

Proof.

Note that:

(i) For $n \in \mathbb{Z}_+$, $a(n) = A(n) - A(n-1)$.

(ii) For $t \in [n, n+1)$ $(n \in \mathbb{Z}_+)$, $A(t) = A(n)$. In particular,

(iii) $A([x]) = A(x)$ and $A([y]) = A(y)$.

So

$$\sum_{y < n \le x} a(n) f(n) \overset{\text{note (i)}}{=} \sum_{n=[y]+1}^{[x]} (A(n) - A(n-1)) f(n)$$

$$= \sum_{n=[y]+1}^{[x]} A(n) f(n) - \sum_{n=[y]}^{[x]-1} A(n) f(n+1)$$

$$= \sum_{n=[y]+1}^{[x]-1} A(n)(f(n) - f(n+1)) + A([x]) f([x]) - A([y]) f([y]+1)$$

Fundamental Theorem of Calculus

$$= -\sum_{n=[y]+1}^{[x]-1} A(n)\int_n^{n+1} f'(t)\,dt + A([x])f([x]) - A([y])f([y]+1)$$

note (ii)

$$= -\sum_{n=[y]+1}^{[x]-1}\int_n^{n+1} A(t)f'(t)\,dt + A([x])f([x]) - A([y])f([y]+1)$$

$$= -\int_{[y]+1}^{[x]} A(t)f'(t)\,dt + A([x])f([x]) - A([y])f([y]+1)$$

Fundamental Theorem of Calculus

$$= -\int_{[y]+1}^{[x]} A(t)f'(t)\,dt - A([x])\left[\int_{[x]}^{x} f'(t)\,dt - f(x)\right] - A([y])\left[\int_y^{[y]+1} f'(t)\,dt + f(y)\right]$$

note (ii)

$$= -\int_{[y]+1}^{[x]} A(t)f'(t)\,dt - \int_{[x]}^{x} A(t)f'(t)\,dt + A([x])f(x) - \int_y^{[y]+1} A(t)f'(t)\,dt - A([y])f(y)$$

note (iii); combine integrals

$$= -\int_y^x A(t)f'(t)\,dt + A(x)f(x) - A(y)f(y).$$

$\square$

Remarks.

1) Putting $a(n)=1$ into ASF gives

$$\sum_{y<n\leq x} f(n) = -\int_y^x [t]f'(t)\,dt + [x]f(x) - [y]f(y)$$

$$= \int_y^x (t-[t])f'(t)\,dt - \int_y^x tf'(t)\,dt$$

$$+[x]f(x)-[y]f(y)$$

integrate by parts

$$= \int_y^x (t-[t])f'(t)\,dt - tf(t)\Big|_y^x + \int_y^x f(t)\,dt$$
$$+[x]f(x)-[y]f(y)$$

$$= \int_y^x (t-[t])f'(t)\,dt - xf(x)+yf(y) + \int_y^x f(t)\,dt$$
$$+[x]f(x)-[y]f(y),$$

which is ESF.

2. Using ASF, we can derive a weaker form of the prime number theorem, as follows.

Let
$$a(n)=\Lambda_1(n)=\begin{cases} \log p & \text{if } p \text{ is prime,} \\ 0 & \text{if not.}\end{cases}$$

Note that, by Theorem 4.10(b) of last time,

$A(x)=\sum_{n\le x} a(n) \le Bx$ for $x\ge 1$, for some $B>0$.

So by ASF, with $f(x)=1/\log x$ and $y=3/2$, and noting that $A(t)=0$ for $t<2$, we have, for $x\ge 3/2$,

$$\sum_{p\le x} 1 = \sum_{n\le x} a(n)f(n)$$
$$= A(x)f(x) - A(3/2)f(3/2) - \int_{3/2}^x A(t)f'(t)\,dt$$
$$= A(x)f(x) - 0\cdot f(3/2) - \int_2^x A(t)f'(t)\,dt$$

$$\le A(x)f(x) + \int_2^x |A(t)f'(t)|\,dt$$

$$\le \frac{Bx}{\log x} + B\int_2^x t\cdot\left|\frac{-1}{t\log^2 t}\right| dt$$

$$\le \frac{Bx}{\log x} + B\int_2^x \frac{dt}{\log^2 t}$$

substitute $t = e^u$ in the integral

$$= \frac{Bx}{\log x} + B\int_{\log 2}^{\log x} \frac{e^u}{u^2}\,du.$$

Since $e^u/u^2 \to \infty$ as $x \to \infty$, $\exists N > 0$ such that, if $x > N$, the maximum of e^u/u^2 on $[\log 2, \log x]$ occurs at $u = \log x$. So for $x > N$,

$$A(x) \le \frac{Bx}{\log x} + \frac{B\cdot e^{\log x}}{\log^2 x}(\log x - 2)$$

$$\le \frac{Bx}{\log x} + \frac{Bx}{\log^2 x}\cdot \log x = \frac{2Bx}{\log x}.$$

This proves that

$$A(x) = \sum_{p \le x} 1 = O\left(\frac{x}{\log x}\right).$$

The prime number theorem is the stronger result that

$$\sum_{p \le x} 1 \sim \frac{x}{\log x}.$$