Shapiro's Tauberian Theorem.

Last time, we showed that

$$\sum_p \left[\frac{x}{p}\right] \log p = x \log x + O(x) \quad \text{for } x \geq 1.$$

We want to "drop the [ ] on the left, then divide through by x." We can do so by:

<u>Thm. 4.8</u> (Shapiro's Tauberian Theorem.)
If a is an arithmetic function, $a(n) \geq 0 \ \forall n$, and

$$\sum_{n \leq x} a(n)\left[\frac{x}{n}\right] = x \log x + O(x) \quad \text{for } x \geq 1,$$

then

(a) $$\sum_{n \leq x} \frac{a(n)}{n} = \log x + O(1) \quad \text{for } x \geq 1.$$

(b) $\exists B > 0$ such that

$$\sum_{n \leq x} a(n) \leq Bx \quad \text{for } x \geq 1.$$

(c) $\exists A > 0$ such that

$$\sum_{n \leq x} a(n) \geq Ax \quad \text{for } x \text{ sufficiently large.}$$

<u>Proof.</u> Write

$$S(x) = \sum_{n \leq x} a(n), \quad T(x) = \sum_{n \leq x} a(n)\left[\frac{x}{n}\right], \quad U(x) = \sum_{n \leq x} \frac{a(n)}{n}.$$

We prove (b) first, by establishing the following claims, under the stated assumptions:

(i) $S(x) - S(x/2) \le T(x) - 2T(x/2)$,
(ii) $T(x) - 2T(x/2) = O(x)$.

Note that (i) and (ii) together imply that, for some $K>0$,

$$S(x) = \sum_{n=0}^{\infty}[S(x/2^n) - S(x/2^{n+1})] \le \sum_{n=0}^{\infty}[T(x/2^n) - 2T(x/2^{n+1})]$$

$$\le \sum_{n=0}^{\infty} Kx/2n \le 2Kx,$$ which is (b).

To prove (i), note that, since $[2y] - 2[y]$ is either 0 or 1, we have $[2y] \ge 2[y]$ for all y, so

$$T(x) - 2T(x/2) = \sum_{n \le x} a(n)\left[\frac{x}{n}\right] - 2\sum_{n \le x/2}\left[\frac{x}{2n}\right]$$

$$= \sum_{n \le x/2} a(n)\left(\left[\frac{x}{n}\right] - 2\left[\frac{x}{2n}\right]\right) + \sum_{\frac{x}{2} < n \le x} a(n)\left[\frac{x}{n}\right]$$

$$\ge \sum_{\frac{x}{2} < n \le x} a(n)\left[\frac{x}{n}\right] = S(x) - S(x/2),$$

which proves (i).

To prove (ii) we note that, since $T(x) = x\log x + O(x)$ by assumption, we have

$$T(x) - 2T(x/2) = x\log x + O(x) - 2\left(\frac{x}{2}\log\frac{x}{2} + O(x)\right)$$

$= x\log x - x(\log x - \log 2) + O(x) = x\log 2 + O(x)$
$= O(x),$

as required. So (b) is proved.

To prove (a), note that $x/n = [x/n] + O(1)$, so

$$U(x) = \sum_{n\le x} \frac{a(n)}{n} = \frac{1}{x}\sum_{n\le x}\frac{x}{n}a(n) = \frac{1}{x}\left(\sum_{n\le x}\left[\frac{x}{n}\right]a(n) + O\left(\sum_{n\le x} a(n)\right)\right)$$

$$= \frac{1}{x}\left(x\log x + O(x) + O(x)\right) = \log x + O(1),$$

the next-to-last equality by assumption and part (b). So (a) is proved.

(c) By part (a),

$$U(x) = \log x + R(x)$$

where $R(x) \le M$ for some $M > 0$, and $x \ge 1$. Suppose $\alpha = e^{-2M-1}$; note that $0 < \alpha < 1$. Then if $\alpha x \ge 1$,

$$\begin{aligned} U(x) - U(\alpha x) &= \log x + R(x) - \log \alpha x - R(\alpha x) \\ &= -\log\alpha + R(x) - R(\alpha x) \\ &\ge -\log\alpha - |R(x)| - |R(\alpha x)| \\ &\ge -\log\alpha - 2M = 1. \end{aligned}$$

But on the other hand,

$$U(x) - U(\alpha x) = \sum_{\alpha x < n \le x} \frac{a(n)}{n} \le \frac{1}{\alpha x}\sum_{n\le x} a(n) = \frac{S(x)}{\alpha x},$$

so

$S(x)/\alpha x \ge 1$ for $\alpha x \ge 1$, which is (c) with $B = \alpha$ and $x \ge 1/\alpha$.

$\square$

Finally, we recall from Thm. 3.16 that, if we define

$$\Lambda_1(n) = \begin{cases} \log p & \text{if } p \text{ is } \underline{\text{prime}}, \\ 0 & \text{if not,} \end{cases}$$

then

$$\sum_{n \le x} \left[\frac{x}{n}\right] \Lambda_1(n) = \sum_{p \le x} \left[\frac{x}{p}\right] \log p = x \log x + O(x),$$

so by Thm. 4.8(a), we have

Thm. 4.10.

(a) $\displaystyle\sum_{p \le x} \frac{\log p}{p} = \sum_{n \le x} \frac{\Lambda_1(n)}{n} = \log x + O(1)$ for $x \ge 1$.

(b) $\displaystyle\sum_{p \le x} \log p \le c_1 x$ for some $c_1 > 0$ and $x \ge 1$.

(c) $\displaystyle\sum_{p \le x} \log p \ge c_2 x$ for some $c_2 > 0$, for x sufficiently large.