Sums of Dirichlet products
(sections 3.10, 3.11).

Goal: to express sums like

$$\sum_{lm \le x} f(l) g(m)$$

in terms of

$$\sum_{n \le x} f(n) \quad \text{and} \quad \sum_{n \le x} g(n).$$

Note:

$$\sum_{lm \le x} f(l) g(m) = \sum_{n \le x} \sum_{lm = n} f(l) g(m)$$

$$= \sum_{n \le x} \sum_{l \mid n} f(l) g(n/l) = \sum_{n \le x} f * g(n),$$

so we're looking at sums of Dirichlet products.

Ultimately, we'll use our results to study

$$\sum_{p \le x} \frac{\log p}{p},$$

which will be relevant to Dirichlet's theorem.

Theorem 3.10

Let f, g be arithmetic functions; let

$$F(x) = \sum_{n \le x} f(n), \qquad G(x) = \sum_{n \le x} g(n) \qquad (x \in \mathbb{R}_+).$$

Then

$$\sum_{n\le x} f*g(n) = \sum_{n\le x} f(n) \sum_{m\le x/n} g(m)$$
$$= \sum_{n\le x} g(n) \sum_{m\le x/n} f(m).$$

Proof.

Write $F(x) = \sum_{m\le x} f(m)$, $G(x) = \sum_{m\le x} g(m)$,

$H(x) = \sum_{n\le x} f*g(n)$.

We wish to show that

$H(x) = \sum_{n\le x} f(n)G(x/n) = \sum_{n\le x} g(n)F(x/n)$; that is,

$H = f\circ G = g\circ F$ where, by defintion,

$k\circ L(x) = \sum_{n\le x} k(n)L(x/n)$, for appropriate k, L.

But, if we define $U(x) = \begin{cases} 0 & \text{if } x<1, \\ 1 & \text{if } x\ge 1, \end{cases}$

then by defintion of H,

$H = (f*g)\circ U = f\circ(g\circ U) = f\circ G$,

the second equality by Thm 2.21. Similarly,

$H = (g*f)\circ U = g\circ(f\circ U) = g\circ F$. $\square$

Now note that, if $g(n) = 1\ \forall n$, then

$G(x) = \sum_{n\le x} g(n) = [x]$,

and

$$f * g(n) = \sum_{d \mid n} f(d).$$

So Thm. 2.10 has this:

<u>Corollary</u> (Thm 3.11).
If
$$F(x) = \sum_{n \le x} f(n),$$
then
$$\sum_{n \le x} \sum_{d \mid n} f(d) = \sum_{n \le x} f(n) \left[\frac{x}{n} \right] = \sum_{n \le x} F(x/n).$$

Application:

<u>Thm 3.12</u>.
For $x \geq 1$:

(a) $$\sum_{n \le x} \mu(n) \left[\frac{x}{n} \right] = 1;$$

(b) $$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n} \right] = \log [x]!$$

<u>Proof</u>.
By Theorem 3.11,

(a) $$\sum_{n \le x} \mu(n) \left[\frac{x}{n} \right] = \sum_{n \le x} \sum_{d \mid n} \mu(d) \overset{\text{Thm. 2.1}}{=} \sum_{n \le x} \left[\frac{1}{n} \right] = 1;$$

(b) $$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n} \right] = \sum_{n \le x} \sum_{d \mid n} \Lambda(d) \underset{\text{Thm. 2.10}}{=} \sum_{n \le x} \log n$$

$$= \log \prod_{n \le x} n = \log [x]! \qquad \square$$

Next:

Thm 3.15.

$$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n} \right] = x \log x - x + O(\log x).$$

Proof.

By Thm. 3.12(b),

$$\sum_{n \le x} \Lambda(n) \left[\frac{x}{n} \right] = \sum_{n \le x} \log n$$

which, by ESF' with $y=1$ and $f(x) = \log x$, equals

$$\int_1^x \log t \, dt + \int_1^x (t - [t])/t \, dt + ([x] - x) \log x$$

$$= x \log x - x + O\left(\int_1^x dt/t \right) + O(\log x)$$

$$= x \log x - x + O(\log x). \qquad \square$$

Finally:

Thm. 3.16.

If $x \ge 2$, then

$$\sum_{p \le x} \left[\frac{x}{p} \right] \log p = x \log x + O(x).$$

Recall that $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha, \\ 0 & \text{if not} \end{cases}$

(p prime, $\alpha \in \mathbb{Z}_+$). So

$[x/n]=0$ for $x<n$

$$\sum_{n\le x}\left[\frac{x}{n}\right]\Lambda(n) = \sum_{n=1}^{\infty}\left[\frac{x}{n}\right]\Lambda(n)$$

$$= \sum_{m=1}^{\infty}\sum_{p}\left[\frac{x}{p^m}\right]\Lambda(p^m) = \sum_{m=1}^{\infty}\sum_{p}\left[\frac{x}{p^m}\right]\log p$$

$$= \sum_{p}\left[\frac{x}{p}\right]\log p + \sum_{m=2}^{\infty}\sum_{p}\left[\frac{x}{p^m}\right]\log p$$

$$= \sum_{p}\left[\frac{x}{p}\right]\log p + \sum_{p}\log p\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]$$

$$\le \sum_{p}\left[\frac{x}{p}\right]\log p + x\sum_{p}\log p\sum_{m=2}^{\infty}p^{-m}$$

$$= \sum_{p}\left[\frac{x}{p}\right]\log p + x\sum_{p}\log p\cdot\frac{1}{p(1-p)}$$

$$= \sum_{p}\left[\frac{x}{p}\right]\log p + O\left(x\sum_{n=2}^{\infty}\frac{\log n}{n^2}\right)$$

$$= \sum_{p}\left[\frac{x}{p}\right]\log p + O(x),$$

because the sum converges by the comparison test and p-series test.
So

$$\sum_{p}\left[\frac{x}{p}\right]\log p = \sum_{n\le x}\Lambda(n)\left[\frac{x}{n}\right] + O(x)$$

which, by Thm. 3.15, equals

$$x\log x - x + O(\log x) + O(x)$$

$$= x\log x + O(x). \qquad \square$$