Some applications of ESF'.

Recall ESF': If f is continuously differentiable on $[y,x]$ $(0<y<x)$, and $y \in \mathbb{Z}_+$, then

$$\sum_{y \le n \le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t-[t])f'(t)\,dt + f(x)([x]-x) + f(y).$$

We have:

Thm. 3.2.

Let $x \geq 1$. Also define Riemann's zeta function $\zeta(s)$ by

$$\zeta(s) = \begin{cases} \sum_{n=1}^{\infty} n^{-s} & \text{if } s>1, \\ \lim_{x\to\infty}\left(\sum_{n\le x} n^{-s} - x^{1-s}/(1-s)\right) & \text{if } 0<s<1. \end{cases}$$

Then

(a) $\sum_{n \le x} 1/n = \log x + C + O(1/x)$,

where C is Euler's constant:

$$C = \lim_{n\to\infty}\left(1+\frac{1}{2}+\dots+\frac{1}{n} - \log n\right).$$

(b) $$\sum_{n\le x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$$ if $s>0$, $s \neq 1$.

(c) $\sum_{n>x} n^{-s} = O(x^{1-s})$ if $s>1$.

(d) $\sum_{n\le x} n^{\alpha} = x^{1+\alpha}/(1+\alpha) + O(x^{\alpha})$ if $\alpha \geq 0$.

Proof.

(a) By ESF$'$ with $f(x) = 1/x$ and $y = 1$,

$$\sum_{n \le x} 1/n = \int_1^x \frac{dt}{t} - \int_1^x (t-[t])t^{-2}\,dt + \frac{[x]-x}{x} + 1$$

$$= \log x - \int_1^\infty (t-[t])t^{-2}\,dt + \int_x^\infty (t-[t])t^{-2}\,dt + O(1/x) + 1.$$

Now $\left|\int_x^\infty (t-[t])t^{-2}\,dt\right| \le \int_x^\infty t^{-2}\,dt = 1/x.$

Also

$$-\int_1^\infty (t-[t])t^{-2}\,dt = -\lim_{n\to\infty}\left(\int_1^n \frac{dt}{t} - \int_1^n \frac{[t]}{t^2}\,dt\right)$$

$$= -\lim_{n\to\infty}\left(\log n - \sum_{k=2}^n \int_{k-1}^k \frac{[t]}{t^2}\,dt\right)$$

$$= -\lim_{n\to\infty}\left(\log n - \sum_{k=2}^n (k-1)\int_{k-1}^k \frac{dt}{t^2}\right)$$

$$= -\lim_{n\to\infty}\left(\log n - \sum_{k=2}^n (k-1)\left(\frac{-1}{k} + \frac{1}{k-1}\right)\right)$$

$$= -\lim_{n\to\infty}\left(\log n - \sum_{k=2}^n 1/k\right) = C - 1.$$

(Note that $\int_1^\infty (t-[t])t^{-2}\,dt$, and hence the limit defining C, converges since $t-[t] = O(1)$.)

So by,

$$\sum_{n \le x} 1/n = \log x + C + O(1/x).$$

(b) By ESF$'$ with $f(x) = x^{-s}$ $(s>0, s\neq 1)$ and $y=1$,

$$\sum_{n\le x} n^{-s} = \int_1^x dt/t^s - s\int_1^x (t-[t])t^{-s-1}\,dt + ([x]-x)x^{-s} + 1$$

$$= \frac{x^{1-s}}{1-s} - \frac{1}{1-s} + 1 + O(x^{-s})$$

$$- s\int_1^\infty (t-[t])t^{-s-1}\,dt + \int_x^\infty (t-[t])t^{-s-1}\,dt, \quad (\sigma)$$

since $[x]-x = O(1)$. (Both integrals converge since $t-[t] = O(1)$ and $s>0$.)

Now, note that the integral over $[x,\infty]$, in (σ), is

$$O\left(\int_x^\infty t^{-s-1}\,dt\right) = O(x^{-s}).$$

So (σ) gives

$$\sum_{n\le x} n^{-s} = \frac{x^{1-s}}{1-s} - \frac{1}{1-s} + 1 - s\int_1^\infty (t-[t])t^{-s-1}\,dt + O(x^{-s})$$

$$= \frac{x^{1-s}}{1-s} + K(s) + O(x^{-s}), \qquad (\lambda)$$

where

$$K(s) = \frac{-1}{1-s} + 1 - s\int_1^\infty (t-[t])t^{-s-1}\,dt.$$

Then by (λ),

$$K(s) = \sum_{n \le x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-s}).$$

Since $K(s)$ is independent of x, we can take the limit as $x \to \infty$, to get

$$K(s) = \lim_{x \to \infty} \left(\sum_{n \le x} n^{-s} - \frac{x^{1-s}}{1-s} \right),$$

which equals $\zeta(s)$ (since, if $s > 1$, then $x^{1-s} \to 0$ and

$$\sum_{n \le x} n^{-s} \to \sum_{n=1}^{\infty} n^{-s} \quad \text{as } x \to \infty). \text{ So}$$

by (λ),

$$\sum_{n \le x} n^{-s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}).$$

(c) If $s > 1$, then

$$\sum_{n > x} n^{-s} = \sum_{n=1}^{\infty} n^{-s} - \sum_{n \le x} n^{-s} = \zeta(s) - \sum_{n \le x} n^{-s}$$
$$= -\frac{x^{1-s}}{1-s} + O(x^{-s}) = O(x^{-s}),$$

the third equality by part (b).

(d) By ESF$'$ with $f(n) = n^{\alpha}$ and $y = 1$ we have, for $\alpha > 0$,

$$\sum_{n\le x} n^\alpha = \int_1^x t^\alpha dt + \alpha\int_1^x (t-[t])t^{\alpha-1}dt + ([x]-x)x^\alpha + 1$$

$$= \frac{x^{1+\alpha}}{1+\alpha} + O\left(\int_1^x t^{\alpha-1}dt\right) + O(x^\alpha)$$

$$= \frac{x^{1+\alpha}}{1+\alpha} + O\left(\frac{x^\alpha}{\alpha} - \frac{1}{\alpha}\right) + O(x^\alpha)$$

$$= \frac{x^{1+\alpha}}{1+\alpha} + O(x^\alpha).$$

On the other hand, if $\alpha = 0$, then

$$\sum_{n\le x} n^\alpha = \sum_{n\le x} 1 = [x] = x + [x] - x$$

$$= x + O(1) = \frac{x^\alpha}{1+\alpha} + O(x^\alpha)$$

as well. $\square$