Part A. The Mangoldt function Λ.

Definition:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(E.g. $\Lambda(1) = 0$, so Λ is not multiplicative.)

Thm. 2.10.
$$\log n = \sum_{d \mid n} \Lambda(d).$$

Proof: $\log 1 = \Lambda(1) = 0$, so the case $n=1$ checks.
Now if $n > 1$, write $n = \prod_{k=1}^{r} p_k^{a_k}$ for distinct primes p_k, and $a_k \in \mathbb{Z}_+$. Since $\Lambda(d) = 0$ unless d is a prime power,

$$\sum_{d \mid n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log p_k$$
$$= \sum_{k=1}^{r} a_k \log p_k = \log\left(\prod_{k=1}^{r} p_k^{a_k}\right)$$
$$= \log(n).$$ $\square$

Also:

Thm. 2.11.
$$\Lambda(n) = \sum_{d \mid n} \mu(d) \log(n/d)$$
$$= -\sum_{d \mid n} \mu(d) \log d.$$

Proof.
The first equality is Möbius inversion

and Thm. 2.10. To prove the second, note that

(i) $\log n = 0$ for $n=1$, while
(ii) for $n>1$,

$$\sum_{d|n} \mu(d) = I(n) = 0, \text{ by Thm. 2.1.}$$

So

$$\sum_{d|n} \mu(d) \log(n/d) = \sum_{d|n} \mu(d) \log n - \sum_{d|n} \mu(d) \log d$$
$$= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d$$
$$= -\sum_{d|n} \mu(d) \log d. \qquad \square$$

Part B. Completely multiplicative functions.

Recall: $f: \mathbb{Z}_+ \to \mathbb{C}$ is completely multiplicative if

$$f(mn) = f(m) f(n) \quad \forall\, m, n \in \mathbb{Z}_+.$$

Examples:
N (given by $N(n) = n$) and u (given by $u(n) = 1$) are completely multiplicative. μ is not, since $\mu(p^2) = 0 \neq \mu(p)^2$ for p prime.

Thm. 2.17.
Suppose f is multiplicative. Then f is completely multiplicative.
$$\iff f^{-1}(n) = \mu(n) f(n) \quad \forall n.$$

Proof.
Suppose f is completely multiplicative, and let $g(n) = \mu(n) f(n)$. Then

$$g*f(n) = \sum_{d|n} \mu(d) f(d) f(n/d)$$

$$= \sum_{d|n} \mu(d) f(n) = f(n) \sum_{d|n} \mu(d)$$

$$= f(n) I(n) = I(n),$$ since $f(1)I(1) = 1$ $= I(1)$, and $f(n)I(n) = 0 = I(n)$ for $n > 1$. So $g = f^{-1}$.

Conversely, suppose f is multiplicative, and $f^{-1}(n) = \mu(n) f(n)$ $\forall n$. Then for p prime and $\alpha > 0$,

$$f(p^\alpha) = -\frac{1}{f^{-1}(1)} \sum_{\substack{d|p^\alpha \\ d<p^\alpha}} f(d) f^{-1}(n/d)$$

$$= -\sum_{k=0}^{\alpha-1} f(p^k) \mu(p^{\alpha-k}) f(p^{\alpha-k})$$

$\mu(p^j) = 0$ for $j > 1$

$$= -f(p^{\alpha-1}) \mu(p) f(p) = f(p^{\alpha-1}) f(p).$$

By an induction argument then, $f(p^\alpha) = f(p)^\alpha$, so f is completely multiplicative. $\square$

Part C. Generalized inversion.

Let $F : \mathbb{R}_+ \to \mathbb{C}$ satisfy $F(x) = 0$ for $x < 1$. Also let $\alpha : \mathbb{Z}_+ \to \mathbb{C}$. We define the generalized convolution $\alpha \circ F$ by

$$\alpha \circ F(x) = \sum_{n \le x} \alpha(n) F(x/n).$$

($\sum_{n \le x}$ always denotes a sum over positive n.)

Note that $\alpha \circ F(x) = 0$ for $x < 1$ as well.

We have:

Thm. 2.21. For $\alpha, \beta : \mathbb{Z}_+ \to \mathbb{C}$ and F as above,

$$\alpha \circ (\beta \circ F) = (\alpha * \beta) \circ F.$$

Proof. For $x \in \mathbb{R}_+$,

$$\alpha \circ (\beta \circ F)(x) = \sum_{n \le x} \alpha(n)\, \beta \circ F(x/n)$$

$$= \sum_{n \le x} \alpha(n) \sum_{m \le x/n} \beta(m) F(x/mn)$$

put $k = mn$

$$= \sum_{nm \le x} \alpha(n)\beta(m) F\left(\frac{x}{mn}\right) = \sum_{k \le x} \left(\sum_{n \mid k} \alpha(n)\beta(k/n) \right) F(x/k)$$

$$= \sum_{k \le x} \alpha * \beta(k) F(x/k) = (\alpha * \beta) \circ F. \quad \square$$

Now note that, for $I(n) = [1/n]$ as before,

$$I \circ F\ (x) = \sum_{n \le x} [1/n] F(x/n) = F(x).$$

Consequently,

Thm. 2.22: Generalized inversion formula. Suppose $\alpha : \mathbb{Z}_+ \to \mathbb{C}$ has Dirichlet inverse α^{-1}. Then

$$G(x) = \sum_{n \le x} \alpha(n) F(x/n) \iff F(x) = \sum_{n \le x} \alpha^{-1}(n) G(x/n).$$

Proof.

$$G = \alpha \circ F \iff \alpha^{-1} \circ G = \alpha^{-1} \circ (\alpha \circ F)$$

Thm. 2.21

$$= (\alpha^{-1} * \alpha) \circ F = I \circ F = F. \qquad \square$$

The above theorem, together with Thm 2.10, yield:

Thm. 2.23: Generalized Möbius inversion.
Suppose α is completely multiplicative.
Then

$$G(x) = \sum_{n \leq x} \alpha(n) F(x/n) \iff F(x) = \sum_{n \leq x} \mu(n) \alpha(n) G(x/n).$$