More on multiplicative functions.

Recall: $f : \mathbb{Z}^+ \to \mathbb{C}$ is multiplicative if $f \not\equiv 0$ and

$$(m,n)=1 \Rightarrow f(mn)=f(m)f(n).$$

Next: if $f(mn)=f(m)f(n)$ for all $m,n \in \mathbb{Z}_+$, then f is completely multiplicative.

Theorem (= Thms. 2.12 - 2.16 combined). Let $f: \mathbb{Z}_+ \to \mathbb{C}$ be multiplicative. Then

(a) $f(1)=1$.

(b) $f(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = f(p_1^{a_1}) f(p_2^{a_2}) \cdots f(p_r^{a_r})$

for all distinct primes p_i and all $a_i \in \mathbb{Z}_+$, and conversely.

(c) f is completely multiplicative iff $f(p^a)=f(p)^a$ for all primes p and $a \in \mathbb{Z}_+$.

(d) g is multiplicative $\Leftrightarrow$ $f * g$ is multiplicative.

(e) f^{-1} is multiplicative, and conversely.

Proof. (a): done. (b,c): straightforward.

d) $\Rightarrow$) Suppose f, g are multiplicative, and $(m,n)=1$. We have

$$f*g(mn) = \sum_{d \mid mn} f(d) g(mn/d).$$

Now note:

(i) If $a|m$ and $b|n$, then $ab|mn$.

(ii) If $d|mn$, write $d = ab$ with $a = (d,m)$ and $b = d/(d,m)$. Then $a|d$. Moreover, Since $d|mn$ we have

$$\frac{d}{(d,m)} \,\Big|\, \frac{mn}{(d,m)};$$

Since

$$\left(\frac{d}{(d,m)}, \frac{m}{(d,m)}\right) = \frac{(d,m)}{(d,m)} = 1,$$

we have $(d/(d,m))|n$ by Thm. 1.5. That is, $b|n$.

(iii) If $a|m$ and $b|n$ we have $(a,b)|(m,n)$, so $(a,b)=1$. Similarly, $(m/a, n/b)=1$.

So positive divisors d of mn are in 1-1 correspondence with products ab, with $a,b \in \mathbb{Z}_+$ and $a|m$, $b|n$. Also, for such a product, we have $(a,b) = (m/a, n/b) = 1$.

So

$$f * g(mn) = \sum_{c|mn} f(c)g\left(\frac{mn}{c}\right) = \sum_{\substack{a|m \\ b|n}} f(ab)g\left(\frac{mn}{ab}\right)$$

$$= \left(\sum_{a|m} f(a)g(m/a)\right)\left(\sum_{b|n} f(b)g(n/b)\right)$$

$$= (f*g(m))(f*g(n)). \qquad \square$$

$\Leftarrow)$ Spz f is multiplicative but g is not. Then $\exists\, m, n \in \mathbb{Z}_+$ with $(m,n)=1$ but $g(mn) \neq g(m)g(n)$. Let (m_0, n_0) be such a pair with minimal product m_0n_0. We wish to show that

$$f*g(m_0n_0) \neq (f*g(m_0))(f*g(n_0)).$$

We do so as follows. By assumption on m_0n_0, we have

$$g(cd) = g(c)g(d) \ \forall c, d \in \mathbb{Z}_+ \text{ with } (c,d)=1 \text{ and } cd < m_0n_0.$$

Then, arguing as in the proof of $\Rightarrow)$, we have

$$f*g(m_0n_0) = \sum_{a|m_0,\, b|n_0} f(ab)\, g\left(\frac{m_0n_0}{ab}\right)$$

$$= \sum_{\substack{a|m_0,\, b|n_0 \\ ab>1}} f(ab)\, g\left(\frac{m_0n_0}{ab}\right) + f(1)g(m_0n_0)$$

by minimality of m_0n_0 ↓

$$= \sum_{\substack{a|m_0,\, b|n_0 \\ ab>1}} f(a)f(b)\, g(m_0/a)\, g(n_0/b) + f(1)g(m_0n_0)$$

$$= \sum_{a|m_0,\, b|n_0} f(a)f(b)\, g(m_0/a)\, g(n_0/b) - g(m_0)g(n_0) + f(1)g(m_0n_0)$$

$$= (f*g(m_0))(f*g(n_0)) - g(m_0)g(n_0) + g(m_0n_0).$$

That is,

$$f * g(m_0 n_0) - (f * g(m_0))(f * g(n_0))$$
$$= g(m_0 n_0) - g(m_0) g(n_0).$$

The right side is nonzero by assumption, so the left side $\neq 0$, so $f * g$ is not multiplicative.

(e) Spz. f is multiplicative. Note that the identity function

$$I = f * f^{-1}$$

is multiplicative. So by part d($\Leftarrow$) above, so is f^{-1}.

Similarly, if f^{-1} is multiplicative, then f is. $\square$