More on Dirichlet products.

Recall: If f and g are arithmetic functions, we define the Dirichlet product $f*g$ ("f splat g") by

$$f*g(n) = \sum_{d|n} f(d)\, g(n/d).$$

Thm 2.6: properties of $*$.

For arithmetic functions f, g, h,

(a) $f*g = g*f$.

(b) $(f*g)*h = f*(g*h)$.

Proof. (a) $f*g(n) = \sum_{d|n} f(d)\, g(n/d)$

define $d' = n/d$: then $d'|n \iff d|n$

$$= \sum_{d'|n} f(n/d')\, g(d') = g*f(n).$$

(b) $$(f*g)*h(n) = \sum_{d|n} f*g(d)\, h(n/d)$$

$$= \sum_{d|n} \sum_{a|d} f(a)\, g(d/a)\, h(n/d). \qquad (⋈☺)$$

Now note that $a \cdot d/a \cdot n/d = n$. Moreover, if $abc = n$ for $a, b, c \in \mathbb{Z}^+$, then, defining $d = n/c$, we have $c = n/d$ and $b = n/ac = n/(a \cdot n/d) = d/a$. So, by (⋈☺),

$$(f*g)*h(n) = \sum_{\substack{a,b,c \in \mathbb{Z}^+ \\ abc = n}} f(a)\, g(b)\, h(c).$$

A similar argument shows that $f*(g*h)(n)$ equals the same. $\square$

We also have:

Thm 2.7. The identity function I, given by $I(n) = [1/n]$, satisfies

$$I * f = f * I = f \quad \forall \text{ arithmetic functions } f.$$

Proof.
$$(f*I)(n) = \sum_{d|n} f(d) I(n/d)$$
$$= \sum_{d|n} f(d) \begin{cases} 1 & \text{if } n/d = 1, \\ 0 & \text{if } n/d > 1 \end{cases}$$

$= f(n)$, which also equals $I*f(n)$ by Thm. 2.6(a). $\square$

Not every arithmetic function f has an inverse under $*$, but:

Thm 2.8. If f is an arithmetic function with $f(1) \neq 0$, then f has a "Dirichlet inverse." That is, $\exists$ a unique arithmetic function f^{-1} with

$$f * f^{-1} = f^{-1} * f = I.$$

Conversely, $f(1) = 0 \iff f$ has no Dirichlet inverse.

Proof Define f^{-1} recursively by

$$f^{-1}(1) = \frac{1}{f(1)},$$

and $f^{-1}(n) = \dfrac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f^{-1}(d) f(n/d)$ if $n > 1$.

Then $f * f^{-1}(1) = \sum_{d|1} f(1) f^{-1}(1) = \dfrac{f(1)}{f(1)} = 1$,

while, for $n > 1$,

$$f^{-1} * f(n) = \sum_{d|n} f^{-1}(d) f(n/d)$$

$$= \sum_{\substack{d|n \\ d<n}} f^{-1}(d) f(n/d) + f^{-1}(n) f(1)$$

$$= -f^{-1}(n) f(1) + f^{-1}(n) f(1) = 0.$$

So $f^{-1} * f(n) = I(n)\ \forall n$.

Uniqueness: spz $g * f = I$. Then

$$(g * f) * f^{-1} = I * f^{-1}$$
$$g * (f * f^{-1}) = f^{-1}$$
$$g * I = f^{-1}$$
$$g = f^{-1}.$$

Conversely, if $f(1) = 0$, then for any arithmetic function g,

$$f * g(1) = \sum_{d|1} f(d) g(1/d) = f(1) g(1) = 0 \neq I(1),$$

so f has no Dirichlet inverse. $\square$

Finally, a super useful result:

<u>Thm. 2.9</u>: Möbius inversion.
For any arithmetic functions f and g,

$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} f(d)\mu(n/d) \quad (⚘)$$

$\forall n$.

<u>Proof</u> Let $u(n) = 1 \ \forall n$, and recall that

$$u * \mu(n) = \mu * u(n) = \sum_{d|n} \mu(d) = I(n) \text{ by Thm. 2.1.}$$

Now the left side of (⚘) says $f = g * u$, which is true iff

$$f * \mu = (g * u) * \mu = g * (u * \mu) = g * I = g,$$

which is true iff the right side of (⚘) holds. $\square$

Note:
Thm 2.2:
$$n = \sum_{d|n} \varphi(d)$$
and Thm. 2.3:
$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

together illustrate Thm. 2.9.