Part A: More properties of $\varphi(n)$.

Thm 2.4

$$\varphi(n) = \begin{cases} n \prod_{p|n} (1 - 1/p) & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases}$$

Proof,

It's clear for $n=1$, so assume $n>1$. Write

$$n = \prod_{i=1}^{r} p_i^{a_i} \quad \text{with } a_i > 0 \text{ for } 1 \le i \le r.$$

By expanding the product,

$$\prod_{p|n} (1 - 1/p) = \prod_{i=1}^{r} (1 - 1/p_i)$$

$$= 1 - \sum_{i=1}^{r} \frac{1}{p_i} + \sum_{\substack{i,j=1 \\ i \ne j}}^{r} \frac{1}{p_i p_j} - \sum_{\substack{i,j,k=1 \\ i \ne j \ne k \ne i}}^{r} \frac{1}{p_i p_j p_k} + \dots + \frac{(-1)^r}{p_1 p_2 \cdots p_r}$$

$$= \sum_{\substack{d|n \\ d \text{ is square-free}}} \frac{(-1)^{\#\text{ of factors of } d}}{d}$$

$$= \sum_{d|n} \mu(d)/d.$$

Now multiply both sides by n, and use Thm. 2.3, which says

$$\sum_{d|n} \mu(d)\, n/d = \varphi(n). \qquad \square$$

Next, we have

Thm. 2.5. Let $m, n, \alpha, a, b \in \mathbb{Z}_+$.

(a) $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$ for p prime.

(b) $$\varphi(mn) = \varphi(m)\varphi(n) \cdot \frac{(m,n)}{\varphi((m,n))}.$$

(c) If $(m,n)=1$, then $\varphi(mn) = \varphi(m)\varphi(n)$.

(d) $a \mid b \Rightarrow \varphi(a) \mid \varphi(b)$.

(e) (i) $\varphi(n)$ is even for $n \geq 3$.
(ii) If r distinct odd primes divide n, for some $r \geq 0$, then $2^r \mid \varphi(n)$.

Proof.
(a) Thm. 2.4 with $n = p^\alpha$.
(b) Note that $p \mid mn \Rightarrow p \mid m$ or $p \mid n$. Also, $p \mid m$ and $p \mid n \Leftrightarrow p \mid (m,n)$.
Then by Thm. 2.4,

$$\frac{\varphi(mn)}{mn} = \prod_{p \mid mn} (1 - 1/p)$$

$$= \frac{\prod_{p \mid m}(1-1/p) \prod_{p \mid n}(1-1/p)}{\prod_{p \mid (m,n)} (1-1/p)}$$

The numerator counts those p with $p \mid m$ and $p \mid n$ twice, so cancel once

$$= \frac{[\varphi(m)/m] \cdot [\varphi(n)/n]}{\varphi((m,n))/(m,n)}.$$

Now multiply through by mn.

(c) By (b) with $(m,n)=1$.

(d) We use induction on b. If $b=1$, then $a|b \Rightarrow a=1$, and the desired result is clear.

Now assume that the result is true for $1 \le b \le k-1$. We wish to deduce that $a|k \Rightarrow \varphi(a)|\varphi(k)$.

This is clear if $a=1$. So assume $a>1$, and $a|k$. Write $k=ac$. By part (b) above,

$$\varphi(k)=\varphi(ac)=\varphi(a)\varphi(c)\cdot\frac{(a,c)}{\varphi((a,c))}. \qquad (*)$$

Now since $a>1$, we have $c<k$; also, $(a,c)|c$, so by induction, $\varphi((a,c))|\varphi(c)$. So by $(*)$,

$$\varphi(k)=\varphi(a)\cdot(a,c)\cdot l \text{ for some } l\in\mathbb{Z}_+.$$

So $\varphi(a)|\varphi(k)$ as required. $\square$

(e) First assume $n=2^\alpha$ for some $\alpha\in\mathbb{Z}_+$. Then by (a), $\varphi(n)=2^\alpha-2^{\alpha-1}=2^{\alpha-1}$! So (e)(i) holds (since $2^\alpha\ge 3 \Rightarrow \alpha\ge 2$), and (e)(ii) holds with $r=0$.

Next, suppose $n=2^\alpha\prod_{i=1}^{r}p_i^{a_i}$ where each p_i is odd, and $\alpha\ge 0$, $a_i>0$ for $1\le i\le r$. By parts (a) and (c) above,

$$\varphi(n)=(2^\alpha-2^{\alpha-1})\prod_{i=1}^{r}(p_i^{a_i}-p_i^{a_i-1}).$$

Each term $p_i^{a_i}-p_i^{a_i-1}$ is even. $\square$

Part B: Dirichlet products (a.k.a. "splat").

Definition: If f and g are arithmetic functions, we define the Dirichlet product $f*g$ ("f splat g") by

$$f*g(n) = \sum_{d|n} f(d)\,g(n/d).$$

For example, if we define the unit function u by $u(n)=1$ $\forall n$, the identity function $I(n)$ by $I(n)=[1/n]$ $\forall n$, and the function N by $N(n)=n$ $\forall n$, then we have

(i) $\mu * u = I$ (Thm. 2.1),
(ii) $\varphi * u = N$ (Thm. 2.2)
(iii) $\mu * N = \varphi$ (Thm. 2.3)

Parts (ii) and (iii) examplify Möbius inversion; see Thm. 2.9 below.