Solutions to Selected Exercises, HW #4

Assignment:

- Section 9.1: Problems 9.1, 9.3. Note: recall that "Find the probability mass function" simply means "Find all possible values of $P(X = x)$."
- Section 9.2: Problems 9.5, 9.6, 9.7. Note: please do Problem 9.6 only for the case $n = 10$. You dont need to do it for general n. Also, for Problem 9.6, the absolute difference between two numbers m and n just means $|m - n|$.
- Section 9.3: Problems 9.14, 9.17. Note: Example 9.6 in the text should be very helpful for doing these two problems.

Problem 9.1. A fair die is tossed two times. Let the random variable X be the largest of the two outcomes. What is the probability mass function of X?

SOLUTION. As usual, we take the sample space S to be

$$S = \{11, 12, \ldots, 16, 21, 22, \ldots, 66\},$$

so that all outcomes are equally likely. The random variable X can take on the values 1 through 6. We have

$$\begin{aligned}
P(X = 1) &= \frac{n(\{11\})}{36} = \frac{1}{36} \approx 2.78\%, \\
P(X = 2) &= \frac{n(\{12, 21, 22\})}{36} = \frac{3}{36} \approx 8.33\%, \\
P(X = 3) &= \frac{n(\{13, 31, 23, 32, 33\})}{36} = \frac{5}{36} \approx 13.89\%, \\
P(X = 4) &= \frac{n(\{14, 41, 24, 42, 34, 43, 44\})}{36} = \frac{7}{36} \approx 19.44\%, \\
P(X = 5) &= \frac{n(\{15, 51, 25, 52, 35, 53, 45, 54, 55\})}{36} = \frac{9}{36} = 25\%, \\
P(X = 6) &= \frac{n(\{16, 61, 26, 62, 36, 63, 46, 64, 56, 65, 66\})}{36} = \frac{11}{36} \approx 30.56\%.
\end{aligned}$$

Problem 9.3. A bag contains three coins. One coin is two-headed and the other two are normal. A coin is chosen at random from the bag and is tossed twice. Let the random variable X denote the number of heads obtained. What is the probability mass function of X?

SOLUTION. The chosen coin is tossed twice, so X, the number of heads, can equal 0, 1, or 2.

What is $P(X = 0)$? Well, to get zero heads, you either choose a normal coin and get zero heads, or you choose the two-headed coin and get zero heads. The probability of choosing a normal coin is $2/3$, and given that you chose a normal coin, the probability of getting zero heads is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The probability of choosing the two-headed coin is $1/3$, and given that you chose the two-headed coin, the probability of getting 0 heads is 0. So

$$P(X = 0) = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 0 = \frac{1}{6} \approx 16.67\%.$$

What is $P(X = 1)$? Well, to get one head, you either choose a normal coin and get one head, or you choose the two-headed coin and get one head. The probability of choosing a normal coin is 2/3, and given that you chose a normal coin, the probability of getting one head is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. (You either get an HT or a TH, out of four equally likely possible outcomes.) The probability of choosing the two-headed coin is 1/3, and given that you chose the two-headed coin, the probability of getting (just) one head is 0. So

$$P(X = 1) = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 0 = \frac{1}{3} \approx 33.33\%.$$

What is $P(X = 2)$? Well, to get two heads, you either choose a normal coin and get two heads, or you choose the two-headed coin and get two heads. The probability of choosing a normal coin is 2/3, and given that you chose a normal coin, the probability of getting two heads is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The probability of choosing the two-headed coin is 1/3, and given that you chose the two-headed coin, the probability of getting two heads is 1. So

$$P(X = 2) = \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot 1 = \frac{1}{2} = 50\%.$$

In sum,

$$P(X = 0) = 16.67\%, \qquad P(X = 1) = 33.33\%, \qquad P(X = 2) = 50\%.$$

Note that the probabilities sum to one, as they should (unless there's roundoff error, in which case you should still get something close to 1).

Problem 9.5. You are playing a game in which four fair dice are rolled. A $1 stake is required. The payoff is $100 if all four dice show the same number and $15 if two dice show the same even number and the other two dice show the same odd number. Is this a fair game? Answer this same question for the following game in which the stake is $2. A fair coin is tossed no more than 5 times. The game ends if the coin comes up tails or five straight heads appear, whichever happens first. You get a payoff of $1 each time heads appears plus a bonus of $25 if five heads appear in a row.

SOLUTION. Consider the first question first. The game is fair if the expected payoff in dollars, call it X, is equal to or better than the stake, which is $1.

What is $E(X)$? Well, the payoff is either $100, $15, or $0. What is $P(X = 100)$; that is, what is the probability that all four dice show the same number? Well, there are 6 different numbers, so there are 6 ways for all four dice to show the same number. The sample space has size $6^4 = 1296$, and all outcomes are equally likely. So

$$P(X = 100) = \frac{6}{1296} \approx 0.0046.$$

Next, what is $P(X = 15)$; that is, what is the probability that two dice show the same even number and the other two dice show the same odd number? Well, there are $\binom{4}{2} = 6$ ways of choosing two dice, and for each such choice, there are 3 different even numbers. Once the first two dice are chosen, the remaining two must show the same odd number, and there are 3 ways for this to happen. So

$$P(X = 15) = \frac{6 \cdot 3 \cdot 3}{1296} \approx 0.0417.$$

So

$$E(X) = 100 \cdot P(X = 100) + 15 \cdot P(X = 15) = 100 \cdot 0.0046 + 15 \cdot 0.0417 = 1.0855.$$

You can expect to get about \$1.09 back on your \$1 stake, so the game is fair. (Or maybe unfair to the dealer.)

Now consider the second question. Denote the payoff by Y. What is $E(Y)$? Well, the payoff is either \$1, \$2, \$3, \$4, \$30, or \$0. What is $P(Y = 1)$? Well, the only way to get exactly one head is if the first coins is heads and the next is tails (because you lose as soon as you get tails). So $P(Y = 1) = 1/4$. What is $P(Y = 2)$? Well, the only way to get exactly two heads is if the first two coins are heads and the thirds is tails. So $P(Y = 2) = 1/8$. Similarly, $P(Y = 3) = 1/16$ (first three are heads and fourth is tails), $P(Y = 4) = 1/32$ (first four are heads and fifth is tails), $P(Y = 30) = 1/32$ (all five are heads). So

$$E(Y) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + 4 \cdot \frac{1}{32} + 30 \cdot \frac{1}{32} = 1.75.$$

The game is not fair, since the stake is \$2.

Problem 9.6. Calculate the expected value of the greater of two numbers when two different numbers are picked at random from the numbers $1, 2, \ldots, n$. What is the expected value of the absolute difference between the two numbers?

SOLUTION. Again, you only need to do this for $n = 10$. In this case its like Problem 9.1 above, but with a ten-sided die. We have

$$\begin{aligned}
P(X = 1) &= \frac{n(\{(1,1)\})}{100} = \frac{1}{100} = 1\%, \\
P(X = 2) &= \frac{n(\{(1,2),(2,1),(2,2)\})}{100} = \frac{3}{100} = 3\%, \\
P(X = 3) &= \frac{5}{100} = 5\%, \\
P(X = 4) &= \frac{7}{100} = 7\%, \\
P(X = 5) &= \frac{9}{100} = 9\%, \\
P(X = 6) &= \frac{11}{100} = 11\%, \\
P(X = 7) &= \frac{13}{100} = 13\%, \\
P(X = 8) &= \frac{15}{100} = 15\%, \\
P(X = 9) &= \frac{17}{100} = 17\%, \\
P(X = 10) &= \frac{19}{100} = 19\%.
\end{aligned}$$

Note that these probabilities do add up to 1. So

$$E(X) = 1 \cdot \frac{1}{100} + 2 \cdot \frac{3}{100} + 3 \cdot \frac{5}{100} + \cdots + 10 \cdot \frac{19}{100} = \frac{143}{20} = 7.15.$$

Now the absolute difference, call it D, between the two numbers can be as small as zero and as large as 9. I don't see any shortcuts here, except to look for patterns.

We have

$$\begin{aligned}
P(D=0) &= \frac{n(\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10)\})}{100} = \frac{10}{100} = 10\%, \\
P(D=1) &= \frac{n\left(\left\{\begin{array}{c}(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4),(5,6),\\(6,5),(6,7),(7,6),(7,8),(8,7),(8,9),(9,8),(9,10),(10,9)\end{array}\right\}\right)}{100} = \frac{18}{100} = 18\%, \\
P(D=2) &= \frac{n\left(\left\{\begin{array}{c}(1,3),(3,1),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4),\\(5,7),(7,5),(6,8),(8,6),(7,9),(9,7),(8,10),(10,8)\end{array}\right\}\right)}{100} = \frac{16}{100} = 16\%, \\
P(D=3) &= \frac{14}{100} = 14\%, \\
P(D=4) &= \frac{12}{100} = 12\%, \\
P(D=5) &= \frac{10}{100} = 10\%, \\
P(D=6) &= \frac{8}{100} = 8\%, \\
P(D=7) &= \frac{6}{100} = 6\%, \\
P(D=8) &= \frac{4}{100} = 4\%, \\
P(D=9) &= \frac{2}{100} = 2\%.
\end{aligned}$$

Again, the probabilities add up to 1. Also,

$$E(D) = 0 \cdot \frac{10}{100} + 1 \cdot \frac{18}{100} + \cdots + 9 \cdot \frac{2}{100} = 3.3.$$

Problem 9.7. You throw darts at a circular target on which two concentric circles of radius 1 cm and 3 cm are drawn. The target itself has a radius of 5 cm. You receive 15 points for hitting the target inside the smaller circle, 8 points for hitting the middle annular region, and 5 points for hitting the outer annular region. The probability of hitting the target at all is 0.75. If the dart hits the target, the hitting point is a completely random point on the target. Let the random variable X denote the number of points thrown on a single throw of the dart. What is the expected value of X?

SOLUTION. The possible values of X are 15, 8, 5, and 0. We have $P(X = 15) = 0.75 \cdot 1/25 = 0.03$, since the bullseye in the center has area $\pi(1)^2 = \pi$ cm, while the entire target has area $\pi(5)^2 = 25\pi$ cm. Also, $P(X = 8) = 0.75 \cdot 8/25 = 0.24$, since the middle annular region has area $\pi(3)^2 - \pi(1)^2 = 8\pi$ cm, while, again, the entire target has area $\pi(5)^2 = 25\pi$ cm. Finally, $P(X = 5) = 0.75 \cdot 16/25 = 0.24$, since the outer annular region has area $\pi(5)^2 - \pi(3)^2 = 16\pi$ cm, while, again, the entire target has area $\pi(5)^2 = 25\pi$ cm. So

$$E(X) = 15 \cdot 0.03 + 8 \cdot 0.24 + 5 \cdot 0.48 = 4.77.$$

Problem 9.14. What is the expected value of the number of times that two adjacent letters will be the same in a random permutation of the eleven letters of the word Mississippi?

SOLUTION. Again, this is very similar to Example 9.6.

Call the number in question X. Also, for $i = 1, 2, \ldots, 10$, let X_i be the random variable that equals 1 if the ith and the $(i+1)$st letters are the same, and 0 otherwise. Then $X = X_1 + X_2 + \ldots + X_{10}$, so $E(X) = E(X_1) + E(X_2) + \cdots + E(X_{10})$.

Now what is $E(X_1)$? Well, $E(X_1)$ only takes the values 1 and 0. What's the probability that $X_1 = 1$? This is the probability that the first two letters are either both i's, both s's, or both p's. (They can't both be m's, since there's only one m.) There are $\binom{4}{2} = 6$ ways for them both to be i's (since there are 4 i's), $\binom{4}{2} = 6$ ways for them both to be s's (since there are 4 s's), and $\binom{2}{2} = 1$ way for them both to be p's (since there are 2 p's). There are $\binom{11}{2} = 55$ total ways to choose 2 letters from the 11, so

$$P(X_1 = 1) = \frac{6+6+1}{55} = \frac{13}{55}.$$

So

$$E(X_1) = 1 \cdot \frac{13}{55} + 0 = \frac{13}{55} \approx 0.2364.$$

Similarly, $E(X_i) = 0.2364$ for each i. So

$$E(X) = 10 \cdot 0.2364 = 2.364.$$

Problem 9.17. What is the expected number of times that two consecutive numbers will show up in a Lotto drawing of six different numbers from the numbers $1, 2, \cdots, 45$?

SOLUTION. This is similar to Example 9.6, and to Problem 9.14 above. Call the number in question Y. Also, for $i = 1, 2, \ldots, 5$, let Y_i be the random variable that equals 1 if the numbers i and $(i+1)$ both show up, and 0 otherwise. Then $Y = Y_1 + Y_2 + \ldots + Y_{44}$, so $E(Y) = E(Y_1) + E(Y_2) + \cdots + E(Y_{44})$.

Now what is $E(Y_1)$? This is the expected number of times that both 1 and 2 show up. There are 6 ways in which a 1 can show up, and assuming a 1 shows up, there are 5 ways in which a 2 can show up, and assuming a 1 and a 2 show up, there are $\binom{43}{4}$ ways of choosing the remaining 4 numbers. Also, there are $\binom{45}{6}$ possible ways of selecting 6 numbers from the 45. So

$$P(Y_1 = 1) = \frac{6 \cdot 5 \cdot \binom{43}{4}}{\binom{45}{6}}.$$

So

$$E(Y_1) = 1 \cdot P(Y_1 = 1) + 0 = \frac{6 \cdot 5 \cdot \binom{43}{4}}{\binom{45}{6}}.$$

Similarly,

$$E(Y_i) = \frac{6 \cdot 5 \cdot \binom{43}{4}}{\binom{45}{6}}$$

for any i. So

$$E(Y) = 44 \cdot \frac{6 \cdot 5 \cdot \binom{43}{4}}{\binom{45}{6}} = \frac{2}{3} \approx 0.6667.$$