Solutions to Selected Exercises, HW #3

Assignment:

- Section 7.3, pages 241–242: Problems 7.28, 7.34, 7.35, 7.36.
- Section 8.1, pages 120–122: Problems 8.2, 8.3, 8.10, 8.11, 8.13.
- Section 8.2, pages 131–134: Problems 8.18, 8.19, 8.20, 8.23.

Problem 7.28. The event A has probability $2/3$ and there is a probability of $3/4$ that at least one of the events A and B occurs. What are the smallest and largest possible values for the probability of event B?

SOLUTION. We have

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Solving for $P(B)$ gives

$$P(B) = P(A \cup B) - P(A) + P(AB).$$

We're given that $P(A) = 2/3$ and $P(A \cup B) = 3/4$, so we get

$$P(B) = \frac{3}{4} - \frac{2}{3} + P(AB) = \frac{1}{12} + P(AB).$$

This will be largest when $P(AB)$ is largest and smallest when $P(AB)$ is smallest.

What's the largest that $P(AB)$ can be? Well, having both A and B happen can't be any more likely than having A happen, so $P(AB) \leq P(A) = 2/3$. So

$$P(B) = \frac{1}{12} + P(AB) \leq \frac{1}{12} + \frac{2}{3} = \frac{3}{4}.$$

The largest that $P(AB)$ can be is $3/4$.

What's the smallest that $P(AB)$ can be? Well it might be zero, since conceivably A and B cannot both happen together. So $P(AB) \geq 0$. So

$$P(B) = \frac{1}{12} + P(AB) \geq \frac{1}{12} + 0 = \frac{1}{12}.$$

The smallest that $P(AB)$ can be is $1/12$.

Problem 7.34. In the Lotto 6/42, six distinct numbers are picked at random from the numbers $1, 2, \ldots, 42$. What is the probability that the number 10 will be picked?

SOLUTION. The number of ways in which 6 numbers can be picked (without regard to order) is $\binom{42}{6}$. The number of ways this can be done *without* any of the numbers being a 10 is $\binom{41}{6}$ (there are 41 numbers that are not equal to 10). So

$$\begin{aligned}
&P(\text{out of six distinct numbers picked, the number 10 is one of them})\\
&= 1 - P(\text{out of six distinct numbers picked, the number 10 is } \textbf{not} \text{ one of them})\\
&= 1 - \frac{\binom{41}{6}}{\binom{42}{6}} \approx 0.1429 = 14.29\%.
\end{aligned}$$

Problem 7.35. Two black socks, two brown socks and one white sock lie mixed up in a drawer. You grab two socks without looking. What is the probability that you have grabbed two black socks or two brown socks?

SOLUTION. Assume that all combinations of socks grabbed are equally likely.

Let A be the event that you grab two black socks, and B the event that you grab two brown socks. Certainly A and B are mutually exclusive, so $P(A \cup B) = P(A) + P(B)$.

Let's compute $P(A)$. There's only one way of grabbing both black socks (if we're not keeping track of order), and there are $\binom{5}{2}$ ways of choosing two socks from the drawer. So

$$P(A) = \frac{1}{\binom{5}{2}} = \frac{1}{10} = 10\%.$$

We get exactly the same number, for exactly the same reason, for $P(B)$. So

$$P(A \cup B) = P(A) + P(B) = 20\%.$$

Problem 7.36. John and Paul play the following game. They each roll one fair die. John wins the game if his score is larger than Paul's score or if the product of the two scores is an odd number. Is this a fair game?

SOLUTION. Presumably, if John doesn't win then Paul does, although the problem doesn't say this specifically. But we'll assume it.

To check whether the game is fair, let's see if $P(\text{John wins}) = 1/2$. Let's write the sample space as

$$S = \{11, 12, \ldots 16, 21, 22, \ldots 66\},$$

where the first number is what John gets and the second is what Paul gets.

The event A where John's number is larger than Paul's is

$$A = \{21, 31, 32, 41, 42, 43, 51, 52, 53, 54, 61, 62, 63, 64, 65.\}$$

Note that $n(A) = 15$. The event B where the product of the two number is odd is

$$B = \{11, 13, 15, 31, 33, 35, 51, 53, 55\}.$$

Note that $n(B) = 9$. The event AB is

$$AB = \{31, 51, 53\},$$

so $n(AB) = 3$. So

$$P(\text{John wins}) = P(A \cup B) = P(A) + P(B) - P(AB) = \frac{15 + 9 - 3}{36} = \frac{21}{36} \approx 0.5833 \neq \frac{1}{2},$$

so the game is not fair.

Problem 8.2. You toss a nickel (5 cents), a dime (10 cents) and a quarter (25 cents), independently of each other. What is the conditional probability that the quarter shows up heads given that the coins showing up heads represent an amount of at least 15 cents?

SOLUTION. Define the sample space S by

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\},$$

where the first letter is how the nickel lands, the second is how the dime lands, and the third is how the quarter lands.

Consider the events

$$A : \text{the quarter shows up heads,}$$
$$B : \text{the coins that show heads add up to at least 15 cents.}$$

By simple counting, we find that

$$P(AB) = P(\{HHH, HTH, THH, TTH\}) = \frac{1}{2};$$
$$P(B) = P(\{HHH, HHT, HTH, THH, TTH\}) = \frac{5}{8}.$$

So

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1/2}{5/8} = 0.8 = 80\%.$$

Problem 8.3. Every evening, two weather stations issue a weather forecast for the next day. The weather forecasts of the two stations are independent of each other. On average, the weather forecast of station 1 is correct in 90% of the cases, irrespective of the weather type. This percentage is 80% for station 2. On a given day, station 1 predicts sunny weather for the next day, whereas station 2 predicts rain. What is the probability that the weather forecast of station 1 will be correct?

SOLUTION. Consider the events

$$A : \text{station 1 is correct,}$$
$$B : \text{station 1 predicts sun and station 2 predicts rain.}$$

We are looking for $P(A|B)$.

What is $P(B)$? Well, the two stations are predicting opposite things, so one must be right and one must be wrong. This can happen in one of two (mutually exclusive) ways: station 1 is right and station 2 is wrong, or station 1 is wrong and station 2 is right. Call the first of these two events C and the second D. Then, since the stations' predictions are independent of each other,

$$P(B) = P(C \cup D) = P(C) + P(D) = 0.9 \cdot 0.2 + 0.1 \cdot 0.8 = 0.18 + 0.08 = 0.26.$$

Also note that $P(AB) = 0.18$, because to say that station 1 is right and station 2 is wrong AND that station 1 predicts sun and station 2 predicts rain is to say that station 1 is right and station 2 is wrong, which has probability $0.9 \cdot 0.2 = 0.18$.

Finally,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.18}{0.26} \approx 0.6923 = 69.23\%.$$

Problem 8.10. You travel from Amsterdam to Sydney with a change of airplanes in Dubai and Singapore. You have one piece of luggage. At each stop your luggage is transferred from one plane to another. At the airport in Amsterdam there is a probability of 5% that your luggage is not placed in the right plane. The probability is 3% at the airport in Dubai and 2% at the airport in Singapore. What is the probability that your luggage does not reach Sydney with you? If your luggage does not reach Sydney with you, what is the probability that it was lost at the airport in Dubai?

SOLUTION. Consider the events

$$\begin{aligned} A &: \text{luggage gets misplaced in Amsterdam,} \\ D &: \text{luggage gets misplaced in Dubai,} \\ S &: \text{luggage gets misplaced in Singapore.} \end{aligned}$$

Presumably, the airports lose luggage independently of each other. So

$$\begin{aligned} &P(\text{luggage doesn't make it to Sydney}) = P(A \cup D \cup S) \\ &= P(A) + P(D) + P(S) - P(AD) - P(AS) - P(DS) + P(ADS) \\ &= P(A) + P(D) + P(S) - P(A)P(D) - P(A)P(S) - P(D)P(S) + P(A)P(D)P(S) \\ &= 0.05 + 0.03 + 0.02 - 0.05 \cdot 0.03 - 0.05 \cdot 0.02 - 0.03 \cdot 0.02 + 0.05 \cdot 0.03 \cdot 0.02 \\ &\approx 0.0969 = 9.69\%. \end{aligned}$$

Morever,

$$\begin{aligned} &P(\text{lost in Dubai, given does not make it}) \\ &= \frac{P(\text{lost in Dubai and does not make it})}{P(\text{does not make it})} = \frac{0.03}{0.0969} \approx 0.309 = 30.9\%. \end{aligned}$$

Problem 8.11. Seven individuals have reserved tickets at the opera. The seats they have been assigned are all in the same row of seven seats. The row of seats is accessible from either end. Assume the seven individuals arrive in a random order, one by one. What is the probability of all seven individuals taking their seats without having to squeeze past an already seated individual?

SOLUTION. To say that a person doesn't have to squeeze past anyone is to say that, once they sit, they are either on the far left or far right of everyone seated.

Clearly the first person seated is either on the far left or far right of everyone seated. So P(first person doesn't squeeze)$= 1$. Similarly, P(second person doesn't squeeze)$= 1$, since with only two people seated, each one is on the far left or far right of everyone seated. Now what about the third person? Well, among three people, there's no reason any one of them should have a greater chance of ending up on the far left than any other. So each of the three people has a $1/3$ chance of ending up on the far left. In particular, the third person to arrive does. Similarly, that person has a $1/3$ chance of ending up on the far right. So $P(\text{third person doesn't squeeze}) = 1/3 + 1/3 = 2/3$.

Next: among four people, there's no reason any one of them should have a greater chance of ending up on the far left than any other. So each of the four people has a $1/4$ chance of ending up on the far left. In particular, the fourth person to arrive does. Similarly, that person has a $1/4$ chance of ending up on the far right. So $P(\text{fourth person doesn't squeeze}) = 1/4 + 1/4 = 2/4$.

Continuing the pattern, we can see that the 5th person has probability $2/5$ of ending up either on the far right or the far left, ...the 7th person has probability $2/7$. Also, note that all of the events $P(k\text{th person doesn't squeeze})$ are independent, since whether or not a person squeezes depends only on where the previous people are, not whether they had to squeeze to get there. So

$$P(\text{no one squeezes}) = 1 \cdot 1 \cdot \frac{2}{3} \cdot \frac{2}{4} \cdot \frac{2}{5} \cdot \frac{2}{6} \cdot \frac{2}{7} \approx 0.0127 = 1.27\%.$$

Problem 8.13. A box contains N identical balls, where one ball has a winning number written on it. In a prespecified order N persons each choose a ball at random from the box. Show that each person has the same probability of getting the winning ball.

SOLUTION. The first person has probability $1/N$ of winning. Also,

$$\begin{aligned}
&P(\text{second wins}) = P(\text{first does not get winning ball and second does}) \\
&= P(\text{first does not get winning ball}) \cdot P(\text{second gets winning ball given that first does not}) \\
&= \frac{N-1}{N} \cdot \frac{1}{N-1} = \frac{1}{N}.
\end{aligned}$$

Next,

$$\begin{aligned}
&P(\text{third wins}) = P(\text{first two get losing balls and third gets winner}) \\
&= P(\text{first two get losing balls}) \cdot P(\text{third gets winning ball given that first two do not}) \\
&= P(\text{first gets a losing ball}) \cdot P(\text{second gets a losing ball given that first does too}) \\
&\cdot P(\text{third gets winning ball given that first two do not}) \\
&= \frac{N-1}{N} \cdot \frac{N-2}{N-1} \cdot \frac{1}{N-2} = \frac{1}{N},
\end{aligned}$$

and so on. The pattern is clear.

Problem 8.18. You have two identical boxes in front of you. One of the boxes contains balls numbered 1 to 10 and the other contains balls numbered 1 to 25. You choose at random one of the boxes and pick a ball at random from the chosen box. What is the probability of picking a ball with the number 7 written on it?

SOLUTION. We saw in class on Friday, 9/22 that, for any events A and B,

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c).$$

Here, B^c just denotes the complement of B, meaning all possible outcomes except those in B. (The above formula is just Rule 8.2 in the book, when there are only two events B_1 and B_2, and $B_2 = B_1^c$.) So let A be the event that a ball with a 7 is drawn, and B the event that the first box (the one with the 10 balls) is chosen. Then

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c) = \frac{1}{2} \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{1}{25} = 0.07 = 7\%.$$

Problem 8.19. A bag contains three coins. One coin is two-headed and the other two are normal. A coin is chosen at random from the bag and is tossed twice. What is the probability of getting two heads? if two heads appear, what is the inverse probability that the two-headed coin was chosen?

SOLUTION. Let A be the event of getting two heads, and B the event that the fair coin was chosen. Then by arguments similar to those in the previous problem,

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c) = \frac{2}{3} \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right) + \frac{1}{3} \cdot (1 \cdot 1) = 0.5 = 50\%.$$

Problem 8.20. In a television game show, the contestant can win a small prize, a medium, prize, and a large prize. The large prize is a sports car. Each of the three prizes is "locked up" in a separate box. There are five keys randomly arranged in front of the contestant. One opens the lock to the small prize, another to the medium prize, another to the large prize. Another is a dud that does not open any of the locks. The final key is the "master key" that opens all three locks. The contestant has a chance to choose up to two keys. For that purpose, the contestant is asked two quiz questions. For each correct answer, they can select one key. The probability of correctly answering any given quiz question is 0.5. The contestant tries the keys they have gained (if any) on all three doors. What is the probability of unlocking the box with the sports car?

SOLUTION. Let A be the event that they get the car, B the event that they get only the first question correct, C the event that they get both questions correct, and D the event that they get only the second correct. Then

$$P(A) = P(B)P(A|B) + P(C)P(A|C) + P(D)P(A|D).$$

Now $P(B) = (0.5)^2 = 0.25$, since each question has a 0.5 probability of being answered correctly. Also, $P(A|B) = 2/5 = 0.4$, since 2 of the 5 keys open the box with the sports car, and the contestant only gets to pick a key if they answer the question correctly.

Next, we have $P(C) = 0.25$. But what is $P(A|C)$? This is the probability of winning given that you answered both questions correctly. But if you answered both correctly, then you win precisely if you pick the right key on the first or second try. Call these events F and S respectively. We have

$$P(F \cup S) = P(F) + P(S) - P(FS) = \frac{2}{5} + \frac{2}{5} - \frac{2}{5} \cdot \frac{1}{5} = \frac{18}{25} = 0.72.$$

Finally, by similar ideas, we have $P(D) = 0.25$ and $P(A|D) = 0.4$. So

$$P(A) = 0.25 \cdot 0.4 + 0.25 \cdot 0.72 + 0.25 \cdot 0.38 = 0.375.$$

Problem 8.23. You repeatedly roll two fair die. What is the probability of two consecutive totals of 7 appearing before a roll with double sixes?

SOLUTION. Let A be the event where you get two consecutive totals of 7 appearing before a roll with double sixes. Notice that there are four different things that can happen on your first two rolls of the two dice.

1. You might get a total of seven on both of your first two rolls. The probability of this happening is $(6/36)(6/36) = 1/36$. (Out of 36 possible outcomes for each roll, only six – namely, the outcomes 16, 25, 34, 43, 52, 61 – give a total of seven.) Note that, in this case, A is guaranteed to happen.

2. One of your first first two rolls might be double sixes (and the other could be anything). The probability of this is $1/36$. And in this case, the probability of A happening is 0.

3. Your first roll might NEITHER total to seven NOR be a double six. The probability of this happening is $29/36$ (since there are 6 ways to total to seven, and one way to be a double six, so there are $36 - 6 - 1 = 29$ ways for neither to happen). But now, if your first roll NEITHER totals to seven NOR is a double six, then the only way A can happen is if, in your REMAINING tosses, you get two consecutive totals of 7 appearing before a roll with double sixes. But the probability of this happening is just $P(A)$ again!

4. Your first roll might total to seven, and your second might NEITHER total to seven NOR be a double six. The probability of the first thing happening is, again, $6/36$, and the probability of the second happening is $29/36$, as noted above. So the probability of both happening is $(6/36)(29/36)$. But now, if your first roll totals to seven and your second NEITHER totals to seven NOR is a double six, then the only way A can happen is if, in your REMAINING tosses, you get two consecutive totals of 7 appearing before a roll with double sixes. But the probability of this happening is just $P(A)$ again!

(It might not be obvious that these are the four things that can happen on your first two rolls, and that they're mutually exclusive. But if you think of each roll as having possible outcome 66 (for a double six), 7 (for the total being 7), or neither (for neither a 66 not a 7), then there are $3 \cdot 3 = 9$ possibilities for the first two rolls. Write down these 9 possibilites – for example, one is (neither, 66) – and check that each fits into one of the four above categories.)

But then, by Rule 8.2 in the book,

$$\begin{aligned} P(A) &= P(\text{category 1})P(A|\text{category 1}) + P(\text{category 2})P(A|\text{category 2}) \\ &\quad + P(\text{category 3})P(A|\text{category 3}) + P(\text{category 4})P(A|\text{category 4}) \\ &= \frac{1}{36} \cdot 1 + \frac{1}{36} \cdot 0 + \frac{29}{36} \cdot P(A) + \frac{6}{36} \cdot \frac{29}{36} \cdot P(A). \end{aligned}$$

We can solve this equation for $P(A)$, to get

$$P(A) = \frac{1/36}{1 - 29/36 - (6/36)(29/36)} = \frac{6}{13} \approx 0.4615 = 46.15\%.$$