Some theorems on functions of a complex variable

Here are some basic things you should know about functions of a single complex variable. We'll assume these results for this course, although references for proofs are given, in case you're interested. Also, you may use any of these theorems in any homework exercises.

Throughout, "domain" refers to an open, connected subset of $\mathbb{C}$. Also, "ucc on D" means "uniformly on any compact subset of D."

Theorem $\mathbb{C}$-1: The principle of analytic/meromorphic continuation. *Suppose f and g are meromorphic on domains D_f and D_g respectively. Suppose that $f = g$ on some set $D \subset D_f \cap D_g$ and that there is at least one accumulation point z_0 of D with $z_0 \in D$. Then $f = g$ on all of $D_f \cap D_g$.*

For a proof, see Nevanlinna, *Introduction to Complex Analysis* (Addison–Wesley, 1964) or any one of numerous other references. The idea is this: f and g are in fact *holomorphic* on the domains D'_f and D'_g obtained by removing from D_f and D_g the poles of f and g respectively. The zeroes of a nonzero holomorphic function on a domain are isolated, but by assumption the zeroes of $f - g$ are not, so $f - g \equiv 0$ on $D'_f \cap D'_g$, and hence on $D_f \cap D_g$. (Or, more precisely, $f - g$ has a removeable singularity at each point of $(D_f \cap D_g) - (D'_f \cap D'_g)$; by defining $(f - g)(z) = 0$ at any such point z, we see that $f - g$ becomes holomorphic, and identically zero, on $D_f \cap D_g$.)

The principle of analytic continuation implies the following: Let f be originally defined and meromorphic on a domain D_f. Suppose there is some function g, meromorphic on some larger domain $E \supset D_f$, that agrees with f on D_f. Then there is only *one* such g, because were there another one, say g_0, then we'd have $g = g_0$ on D_f, and hence on E by analytic continuation.

If there *is* such an extension g of a function f, meromorphic on D_f, to a larger domain, then g is called the *(meromorphic) continuation* of f. It's customary to denote a function and its continuation (to a given larger domain) by the same letter.

Theorem $\mathbb{C}$-2: Weierstrass' theorem on uniformly convergent sequences of holomorphic functions. *If $f_n(s)$ is holomorphic on a domain $D \subset \mathbb{C}$ for each $n \in \mathbb{Z}^+$, and $f_n(s) \to f(s)$ ucc on D, then $f(s)$ is holomorphic on D, and moreover the kth derivative $f_n^{(k)}(s)$ converges ucc on D to $f^{(k)}(s)$ for any $k \geq 0$.*

For a proof, see Markushevich, *Theory of Functions of a Complex Variable Vol. I* (Prentice Hall, 1965), p. 330.

Corollary $\mathbb{C}$-Σ. *If $f_n(s)$ is holomorphic on a domain $D \subset \mathbb{C}$ for each $n \in \mathbb{Z}^+$, and*

$$\sum_{n=1}^{\infty} f_n(s)$$

converges ucc on D, then this sum defines a function $h(s)$ analytic on D, and moreover

$$h'(s) = \sum_{n=1}^{\infty} f'_n(s)$$

(and similarly for higher derivatives).

Theorem $\mathbb{C}$-3: Differentiation under the integral sign. *Suppose I is an interval in $\mathbb{R}$, D is a domain in $\mathbb{C}$, $|\varphi(t,s)|$ is integrable on I for each $s \in D$, $\partial\varphi(t,s)/\partial s$ exists for all $(t,s) \in I \times D$, and there exists a function h that is integrable on I and such that, for each $s \in D$, $\left|\partial\varphi(t,s)/\partial s\right| \leq h(t)$. Then*

$$g(s) = \int_I \varphi(t,s)\, d\omega$$

is holomorphic on D, and moreover

$$g'(s) = \int_I \frac{\partial}{\partial s}\varphi(t,s)\, d\omega$$

for $s \in D$.

For a proof of a similar theorem (where instead of D we have some interval $[c,d]$), see Marsden, *Elementary Classical Analysis* (Freeman, 1974), p. 374. It's not too hard to tweak Marsden's proof, using the Lebesgue Dominated Convergence Theorem, to get the result just stated.

Theorem $\mathbb{C}$-4: Analyticity of infinite products. *Let f_n be analytic on a domain D for each $n \in \mathbb{Z}^+$, and suppose f_n does not identically equal -1 for any n. If $\sum_{n=1}^{\infty} f_n(s)$ converges absolutely and ucc on D, then*

$$\prod_{n=1}^{\infty}(1 + f_n(s))$$

converges absolutely to a function $g(s)$ analytic on D. If $g(a) = 0$ for some $a \in D$ then $f_n(a) = -1$ for only finitely many $n \in \mathbb{Z}^+$; moreover in this case the multiplicity of the zero of g at $s = a$ is the sum over n of the multiplicities of the zeroes of the functions $1 + f_n$ at $s = a$.

For a proof, see Conway, *Functions of a Complex Variable*, 2nd Ed. (Springer–Verlag, 1978), p. 167.