Part A. The Mangoldt function Λ.

Definition:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(E.g. $\Lambda(1) = 0$, so Λ is not multiplicative.)

Thm. 2.10.

$$\log n = \sum_{d \mid n} \Lambda(d).$$

Proof: $\log 1 = \Lambda(1) = 0$, so the case $n = 1$ checks.

Now if $n > 1$, write $n = \prod_{k=1}^{r} p_k^{a_k}$ for distinct primes p_k, and $a_k \in \mathbb{Z}_+$. Since $\Lambda(d) = 0$ unless d is a prime power,

$$\sum_{d \mid n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda(p_k^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log p_k$$

$$= \sum_{k=1}^{r} a_k \log p_k = \log\left(\prod_{k=1}^{r} p_k^{a_k} \right)$$

$$= \log(n). \qquad \square$$

Also:

Thm. 2.11.

$$\Lambda(n) = \sum_{d \mid n} \mu(d) \log(n/d)$$

$$= -\sum_{d \mid n} \mu(d) \log d.$$

Proof.

The first equality is Möbius inversion

and Thm. 2.10. To prove the second, note that

(i) $\log n = 0$ for $n=1$, while
(ii) for $n>1$,

$$\sum_{d|n} \mu(d) = I(n) = 0, \text{ by Thm. 2.1.}$$

So

$$\sum_{d|n} \mu(d)\log(n/d) = \sum_{d|n} \mu(d)\log n - \sum_{d|n} \mu(d)\log d$$

$$= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d)\log d$$

$$= -\sum_{d|n} \mu(d)\log d. \qquad \square$$

Part B. Completely multiplicative functions.

Recall: $f: \mathbb{Z}_+ \to \mathbb{C}$ is completely multiplicative if

$$f(mn) = f(m)f(n) \quad \forall\, m, n \in \mathbb{Z}_+.$$

Thm. 2.17.
Suppose f is multiplicative. Then f is completely multiplicative
$$\iff f^{-1}(n) = \mu(n)f(n) \quad \forall n.$$

Proof.
Suppose f is completely multiplicative, and let $g(n) = \mu(n)f(n)$. Then, for $n \in \mathbb{Z}^+$,

$$g * f(n) = \sum_{d|n} \mu(d)f(d)f(n/d)$$

$$= \sum_{d|n} \mu(d)f(n) = f(n)\sum_{d|n} \mu(d)$$

$= f(n)I(n) = I(n)$, since $f(1)I(1) = 1$
$= I(1)$, and $f(n)I(n) = 0 = I(n)$ for $n > 1$. So
$g = f^{-1}$.

Conversely, suppose f is multiplicative, and $f^{-1}(n) = \mu(n)f(n)\ \forall n$. Then for p prime and $\alpha \in \mathbb{Z}_+$, the fact that $f * f^{-1}(p^\alpha) = I(p^\alpha) = 0$ implies

$$0 = \sum_{d|p^\alpha} f^{-1}(d) f(p^\alpha/d) = \sum_{k=0}^{\alpha} f^{-1}(p^k) f(p^{\alpha-k})$$

$$= \sum_{k=0}^{\alpha} \mu(p^k) f(p^k) f(p^{\alpha-k})$$

$\mu(p^k) = 0$ for $k > 1$

$$= \mu(1)f(1)f(p^\alpha) + \mu(p)f(p)f(p^{\alpha-1})$$
$$= f(p^\alpha) - f(p)f(p^{\alpha-1}),$$

so $f(p^\alpha) = f(p)f(p^{\alpha-1})$. By induction, $f(p^\alpha) = f(p)^\alpha$, so by the thm from last time, f is completely multiplicative. □

Part C: Example. Let f be the multiplicative af defined by

$$f(p^\alpha) = \begin{cases} -1-p & \text{if } \alpha = 1, \\ p & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha > 2 \end{cases}$$

(p prime, $\alpha \in \mathbb{Z}^+$).

Compute $f^{-1}(n)$ for $n = 1, 2, 3, \ldots, 7$ and $n = 14$, $n = 28$. Conjecture and prove: what is f^{-1}?

Solution.

Using Thm. 2.8:

$f^{-1}(1) = 1$
$f^{-1}(2) = 3$
$f^{-1}(3) = 4$
$f^{-1}(4) = 7$
$f^{-1}(5) = 6$
$f^{-1}(6) = 12$
$f^{-1}(7) = 8$
$f^{-1}(14) = 24$
$f^{-1}(28) = 56$

Conjecture.

f^{-1} is the divisor function

$$\sigma(n) = \sum_{d|n} d.$$

Proof. See HW #3.

Thm. 2.21

$= (\alpha^{-1} * \alpha) \circ F = I \circ F = F.$ □

The above theorem, together with Thm 2.10, yield:

Thm. 2.23: Generalized Möbius inversion. Suppose α is completely multiplicative. Then

$$G(x) = \sum_{n \le x} \alpha(n) F(x/n) \iff F(x) = \sum_{n \le x} \mu(n) \alpha(n) G(x/n).$$