Multiplicative functions.

Throughout, f, g are af's; $m, n, d, k, l \in \mathbb{Z}_+$; $p, p_1, p_2, \ldots$ are primes; $a, a_1, a_2, \ldots \in \mathbb{Z}_{\geq 0}$.

Definition.
f is <u>multiplicative</u> if f is not the zero function and

$$(m,n)=1 \Rightarrow f(mn)=f(m)f(n).$$

If $f \not\equiv 0$ and $f(mn)=f(m)f(n)$ for <u>all</u> m, n, then f is <u>completely multiplicative</u>.

Examples.
N, u, I, φ, μ are multiplicative; N, u, I are completely multiplicative.

Theorem (= Thms. 2.12-2.16)
If f is multiplicative, then

(a) $f(1)=1$,

(b) $f(p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}) = f(p_1^{a_1}) f(p_2^{a_2}) \cdots f(p_r^{a_r})$

for distinct $p_1, p_2, \ldots, p_r$.

(c) f is completely multiplicative iff $f(p^a)=f(p)^a$ $\forall$ primes p, $a \in \mathbb{Z}_+$.

(d) g is multiplicative $\Leftrightarrow$ $f*g$ is.

(e) f^{-1} is multiplicative, and conversely.

Proof.

(a) Let n be such that $f(n) \neq 0$. Then
$$f(n) = f(n \cdot 1) = f(n)\cdot f(1);$$
now divide by $f(1)$.

(b) induction on r.

(c) follows from (b).

(d) Let f, g be multiplicative; assume $(m,n)=1$. Note that

$$\{d : d \mid mn\} = \{\text{products } k\ell : k \mid m,\ \ell \mid n\}.$$

Moreover, for such k and ℓ, $(k,\ell)=1$ and $\left(\frac{m}{k}, \frac{n}{\ell}\right) = 1$, since any divisor of both k and ℓ (respectively, both m/k and n/ℓ) divides m and n. So

$$\begin{aligned}
f * g(mn) &= \sum_{d \mid n} f(d)\, g(mn/d) \\
&= \sum_{k \mid m} \sum_{\ell \mid n} f(k\ell)\, g\left(\frac{mn}{k\ell}\right) \\
&= \sum_{k \mid m} \sum_{\ell \mid n} f(k) f(\ell)\, g\left(\frac{m}{k}\right) g\left(\frac{n}{\ell}\right) \\
&= \left[\sum_{k \mid m} f(k)\, g\left(\frac{m}{k}\right)\right]\left[\sum_{\ell \mid m} f(\ell)\, g\left(\frac{n}{\ell}\right)\right] \\
&= (f * g(m))(f * g(n)).
\end{aligned}$$

Before proving the other direction of (d), we prove (e).

By definition of f^{-1}, we have $f^{-1}(1) = 1/f(1) = 1$, whence $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$ if either m or n equals 1.

Now assume $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$ for $(m,n)=1$, $m,n>1$, and $1 \le mn \le s-1$ for some $s \in \mathbb{Z}_+$. Suppose $(m,n)=1$ and $mn=s$. Note that

$$\{d : d \mid mn \text{ and } d<s\}$$
$$= \{\text{products } k\ell : k \mid m \text{ and } k<m, \ \ell \mid n \text{ and } \ell<n\}$$
$$\cup \{\text{products } m\ell : \ell \mid n \text{ and } \ell < n\}$$
$$\cup \{\text{products } kn : k \mid m \text{ and } k<m\} \quad (\text{disjoint union}).$$

So, since $f(1)=1$,

$$f^{-1}(mn) = \frac{-1}{f(1)} \sum_{\substack{d \mid mn \\ d<mn}} f^{-1}(d) f\left(\frac{mn}{d}\right)$$

$$= -\sum_{\substack{k \mid m,\, k<m \\ \ell \mid n,\, \ell<n}} f^{-1}(k\ell) f\left(\frac{mn}{k\ell}\right) - \sum_{\ell \mid n,\, \ell<n} f^{-1}(m\ell) f\left(\frac{mn}{m\ell}\right) - \sum_{k \mid m,\, k<m} f^{-1}(kn) f\left(\frac{mn}{kn}\right)$$

$$\overset{\text{induction hypothesis}}{=} -\left[\sum_{k \mid m,\, k<m} f^{-1}(k) f\left(\frac{m}{k}\right)\right]\left[\sum_{\ell \mid n,\, \ell<n} f^{-1}(\ell) f\left(\frac{n}{\ell}\right)\right]$$
$$- f^{-1}(m) \sum_{\ell \mid n,\, \ell<n} f^{-1}(\ell) f\left(\frac{n}{\ell}\right) - f^{-1}(n) \sum_{k \mid m,\, k<m} f^{-1}(k) f\left(\frac{m}{k}\right)$$

$$= -f^{-1}(m)f^{-1}(n) - f^{-1}(m)[-f^{-1}(n)] - f^{-1}(n)[-f^{-1}(m)]$$
$$= f^{-1}(m)f^{-1}(n).$$

So f^{-1} is multiplicative by strong induction.

To finish (d), note that

$\underline{g} = g*(f*f^{-1}) = (g*f)*f^{-1}$, which is multiplicative if f and $f*g$ are. $\square$