Monday, 8/31

More on Dirichlet products.

Recall: If f and g are arithmetic functions, we define the Dirichlet product $f*g$ ("f splat g") by

$$f*g(n) = \sum_{d \mid n} f(d)\,g(n/d).$$

Thm 2.6: properties of $*$. For arithmetic functions f, g, h,

(a) $f*g = g*f$.

(b) $(f*g)*h = f*(g*h)$.

Proof. (a) See HW #2.

(b)
$$(f*g)*h(n) = \sum_{d \mid n} [f*g(d)]\,h(n/d)$$
$$= \sum_{d \mid n} \sum_{a \mid d} f(a)\,g(d/a)\,h(n/d). \quad (⋈☺)$$

Now note that $a \cdot d/a \cdot n/d = n$. Moreover, if $abc = n$ for $a, b, c \in \mathbb{Z}^+$, then, defining $d = n/c$, we have $c = n/d$ and $b = n/ac = n/(a \cdot n/d) = d/a$. So, by (⋈☺),

$$(f*g)*h(n) = \sum_{\substack{a,b,c \in \mathbb{Z}^+ \\ abc = n}} f(a)\,g(b)\,h(c).$$

A similar argument shows that $f*(g*h)(n)$ equals the same. $\square$

Thm 2.7.

The identity function I, given by $I(n) = [1/n]$, satisfies

$$I * f = f * I = f \quad \forall \text{ arithmetic functions } f.$$

Proof. See HW #2. $\square$

Also:

Thm. 2.8.

Let f be an af. Then $f(1) \neq 0 \iff f$ has a "Dirichlet inverse," meaning $\exists$ an af f^{-1} with

$$f * f^{-1} = f^{-1} * f = I.$$

If $f(1) \neq 0$, then f^{-1} is unique, and defined recursively by

$$f^{-1}(n) = \begin{cases} 1/f(1) & \text{if } n = 1, \\ \dfrac{-1}{f(1)} \displaystyle\sum_{\substack{d \mid n \\ d < n}} f^{-1}(d) f(n/d) & \text{if } n > 1. \end{cases}$$

Proof. If $f(1) = 0$, then $\nexists\, g : \mathbb{Z}_+ \to \mathbb{C}$ such that

$$f * g(1) = f(1) g(1) = I[1] = 1.$$

So suppose $f(1) \neq 0$: define f^{-1} as above. Then $f * f^{-1}(1) = f(1) \cdot 1/f(1) = 1 = I[1]$; moreover, for $n > 1$,

$$f * f^{-1}(n) = f^{-1} * f(n) = \sum_{d \mid n} f^{-1}(d) f(n/d)$$

break off $d=n$ term

$$\overset{\swarrow}{=} \; f^{-1}(n)f(1) + \sum_{\substack{d\mid n \\ d<n}} f^{-1}(d)\,f(n/d)$$

$$= f^{-1}(n)f(1) - f(1)f^{-1}(n) = 0 = I(n).$$

Uniqueness: If $g*f = I$, then

$$g = g*I = g*(f*f^{-1}) = (g*f)*f^{-1} = I*f^{-1} = f^{-1}.$$

$\square$

Finally, a super useful result:

<u>Thm. 2.9</u>: Möbius inversion.
For any arithmetic functions f and g,

$$f(n) = \sum_{d\mid n} g(d) \iff g(n) = \sum_{d\mid n} f(d)\,\mu(n/d) \qquad (⚘)$$

$\forall n.$

<u>Proof</u> Let $u(n) = 1$ $\forall n$, and recall that

$$u*\mu(n) = \mu*u(n) = \sum_{d\mid n} \mu(d) = I(n) \text{ by Thm. 2.1.}$$

Now the left side of (⚘) says $f = g*u$, which is true iff

$$f*\mu = (g*u)*\mu = g*(u*\mu) = g*I = g,$$

which is true iff the right side of (⚘) holds.

$\square$

<u>Note</u>. Thm. 2.9 is exemplified by Thms. 2.2 and 2.3:

$$N = \varphi * u \qquad \text{and} \qquad \varphi = \mu * N.$$