Part A: More properties of $\varphi(n)$.

Thm. 2.4. We have $\varphi(1)=1$, and

$$\varphi(n) = n \prod_{p \mid n} (1-1/p) \quad \text{if } n>1.$$

Proof

The case $n=1$ is clear, so let $n>1$ and write $n = \prod_{i=1}^{r} p_i^{a_i}$ with each $a_i>0$. Then

$$n \prod_{p \mid n} (1-1/p) = n\left[(1-1/p_1)(1-1/p_2)\cdots(1-1/p_r)\right]$$

$$= n\left[1 - \sum_{i=1}^{r} \frac{1}{p_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{r} \frac{1}{p_i p_j} - \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{r} \frac{1}{p_i p_j p_k} + \cdots + \frac{(-1)^r}{p_1 p_2 \cdots p_r}\right]$$

$$= n \sum_{\substack{d \mid n \\ d \text{ squarefree}}} \frac{(-1)^{\#\text{ of factors of } d}}{d} = n \sum_{d \mid n} \frac{\mu(d)}{d}$$

$$= \varphi(n),$$

by Thm. 2.3. □

Next:

Thm. 2.5. Let $m, n, a, b, \alpha \in \mathbb{Z}_+$, and $r \in \mathbb{Z}_{\geq 0}$.

(a) $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$ for p prime.

(b) $\varphi(mn) = \varphi(m)\varphi(n) \cdot \dfrac{(m,n)}{\varphi((m,n))}$; in particular

(c) $(m,n)=1 \Rightarrow \varphi(mn) = \varphi(m)\varphi(n)$.

(d) $a \mid b \Rightarrow \varphi(a) \mid \varphi(b)$.

(e) $\varphi(n)$ is even for $n \geq 3$. Moreover, if s distinct odd primes divide n, then $2^s \mid \varphi(n)$.

Proof.

(a) Thm. 2.4 with $n = p^\alpha$.

(b,c) Note that $p \mid mn \Rightarrow p \mid m$ or $p \mid n$, and that $p \mid m$ and $p \mid n \Leftrightarrow p \mid (m,n)$. So by Thm. 2.4,

$$\varphi(m)\varphi(n) \cdot \frac{(m,n)}{\varphi((m,n))} = \frac{\varphi(m)\varphi(n)}{\prod_{p \mid (m,n)} (1 - 1/p)}$$

$$= \frac{1}{mn} \cdot \frac{\left[\prod_{p \mid m} (1 - 1/p)\right]\left[\prod_{p \mid n} (1 - 1/p)\right]}{\prod_{p \mid (m,n)} (1 - 1/p)}$$

denominator cancels double-counted numerator factors

$$= \frac{1}{mn} \prod_{p \mid mn} (1 - 1/p) = \varphi(mn). \qquad \square$$

(d) Induction on b: since $b = 1$ and $a \mid b \Rightarrow a = 1$, the base case $a = 1$ holds. So assume the result for $1 \leq b \leq k-1$. We show that $a \mid k \Rightarrow \varphi(a) \mid \varphi(k)$: the case $a = 1$ is clear. So assume $a > 1$ and $a \mid k$. Write $k = ac$: then by part (b) above,

$$\varphi(k) = \varphi(a)\varphi(c) \cdot \frac{(a,c)}{\varphi((a,c))}. \qquad (*)$$

But $a > 1$ so $c < k$. So, since $(a,c) \mid c$, we have $\varphi((a,c)) \mid \varphi(c)$, by the induction hypothesis.

So by (*),

$$\varphi(k) = \varphi(a) \cdot l \cdot (a,c)$$

for some $l \in \mathbb{Z}$. So the induction step holds.

(e) Let $n \geq 3$. Two cases:

(i) $n = 2^\alpha$ for some $\alpha \in \mathbb{Z}_+$. Then $\varphi(n) = 2^\alpha - 2^{\alpha-1} = 2^{\alpha-1}$ by (a), and since $n \geq 3$, $\alpha \geq 2$, so $\varphi(n)$ is even.

(ii) n has $s \geq 1$ distinct odd prime power divisors $p_i^{a_i}$. Then by part (c) above, $\varphi(n)$ is divisible by

$$\prod_{i=1}^{s} \varphi(p_i^{a_i}).$$

But each such factor is even, since

$$\varphi(p^a) = p^a - p^{a-1} = \text{odd} - \text{odd} = \text{even} \quad \text{for } p \text{ odd}.$$

So $2^s \mid \varphi(n)$. □

Part B: Dirichlet products

Definition: for af's f, g, define the **Dirichlet product** $f * g$ ("f splat g") by

$$f * g(n) = \sum_{d \mid n} f(d) g(n/d).$$

For example, define, for $n \in \mathbb{Z}_+$,

$u(n) = 1$ (unit function);
$I(n) = [1/n]$ (identity function);
$N(n) = n$ (N function).

Then, by Thms. 2.1, 2.2, 2.3 respectively:

(i) $\mu * u = I$
(ii) $\varphi * u = N$
(iii) $\mu * N = \varphi$.