Arithmetic functions

An arithmetic function (af) is a function $f: \mathbb{Z}_+ \to \mathbb{C}$. We consider af's under the usual $+$, and the "Dirichlet product" $*$ (defined next time).

Conventions:

(i) $\sum_{d|n}$ means sum over positive divisors d of n.

(ii) For $n > 1$, write $n = \prod_{i=1}^{r} p_i^{a_i}$ if $p_1, p_2, \ldots, p_r$ are the distinct prime divisors of n. (So tacitly, each $a_i \in \mathbb{Z}_+$.)

Example 1.
The Möbius function μ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if, in note (ii) above, each } a_i = 1; \\ 0 & \text{otherwise.} \end{cases} \quad *$$

* In this case, n is squarefree.

Now define $[x]$ = the greatest integer $\leq x$, for $x \in \mathbb{R}$. We have:

Thm 2.1 For $n \in \mathbb{Z}_+$,

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.
The case $n=1$ is clear, so suppose $n>1$.

If $d \mid n$, then $\mu(d) \neq 0 \iff d=1$, or d is a product of exactly j of the r distinct prime divisors of n, for some $1 \leq j \leq r$. For a given j, there are $\binom{r}{j}$ such products. So

$$\sum_{d \mid n} \mu(d) = 1 + \binom{r}{1}(-1) + \binom{r}{2}(-1)^2 + \dots + \binom{r}{r}(-1)^r$$

$$= (1+(-1))^r = 0. \quad \square$$

(binomial theorem)

Example 2: Euler's "totient" or "phi" function.

Definition. $\varphi(d) =$ # of positive integers k with $1 \leq k \leq d$ and $(k,d)=1$.

Theorem 2.2.

$$\sum_{d \mid n} \varphi(d) = n.$$

Proof. Let $S = \{1/n, 2/n, 3/n, \dots, n/n\}$. Write each element of S in reduced form: a denominator d will arise iff $d \mid n$ and $d>0$. Moreover, each such d arises $\varphi(d)$ times, since a fraction k/d will arise iff $1 \leq k \leq d$ and $(k,d)=1$. No two reduced fractions are the same since S comprises distinct elements. So

$$\sum_{d \mid n} \varphi(d) = |S| = n. \quad \square$$

There's a nice relation between μ and φ:

Theorem 2.3.

$$\sum_{d \mid n} \mu(d) \cdot \frac{n}{d} = \varphi(n).$$

Proof.

Rewrite the definition of $\varphi(n)$ in the form

$$\varphi(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} 1; \quad \text{equivalently,}$$

$$\varphi(n) = \sum_{k=1}^{n} \left[\frac{1}{(k,n)} \right].$$

By Theorem 2.1, then,

$$\varphi(n) = \sum_{k=1}^{n} \sum_{d \mid (n,k)} \mu(d) = \sum_{k=1}^{n} \sum_{\substack{d \mid n \\ d \mid k}} \mu(d). \qquad (*)$$

Note that a summand $\mu(d)$ appears on the right side of $(*)$ exactly for the integers $d > 0$ such that:

(a) $d \mid n$, and
(b) $\exists\, k \in \mathbb{Z}_+$ with $1 \leq k \leq n$ and $d \mid k$.

But the elements of $\mathbb{Z}_+$ that are $\leq n$ and are divisible by a given d dividing n are precisely the elements

$$d, 2d, \ldots \left(\frac{n}{d}\right) \cdot d.$$

There are n/d such elements. So by $(*)$

$$\varphi(n) = \sum_{d \mid n} \mu(d) \cdot \frac{n}{d}. \qquad \square$$

Remark.

Thms. 2.2 and 2.3 together exemplify Mobius inversion in the ring of af's. More next time.