Some Poisson Random Variable Review Problems

Recall the following. Suppose a certain event happens, on average, λ times per interval of a given extent. Let X be the number of times that the event *actually* happens in such an interval. Then we say "X is Poisson of parameter λ," and we have the formula

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad (k = 0, 1, 2, 3, \ldots). \qquad (*)$$

1. On a given day, the Happa-One ski resort in Hakuba, Japan opens additional lifts depending on how many buses from Tokyo arrive at the resort by 11 AM. If fewer than 3 buses arrive by 11 AM, they open no additional lifts. If 3 or 4 buses arrive by 11 AM, they open one additional lift. If 5 or more arrive by 11 AM, they open two additional lifts.

   Suppose, on average, 2 buses from Tokyo arrive at the resort by 11 AM.

   (a) Find the probability that, on a given day, Happa-One will open no additional lifts. Hint: this probability equals $P(X = 0) + P(X = 1) + P(X = 2)$, where X is the number of buses arriving by 11 from Tokyo.

   $$P(\text{no additional lifts}) = P(X = 0) + P(X = 1) + P(X = 2)$$
   $$= \frac{2^0}{0!}e^{-2} + \frac{2^1}{1!}e^{-2} + \frac{2^2}{2!}e^{-2} = 0.6767.$$

   (b) Find the probability that, on a given day, Happa-One will open one additional lift.

   $$P(\text{one additional lift})$$
   $$= P(X = 3) + P(X = 4) = \frac{2^3}{3!}e^{-2} + \frac{2^4}{4!}e^{-2} = 0.2707.$$

   (c) Find the probability that, on a given day, Happa-One will open two additional lifts.

   $$P(\text{two additional lifts})$$
   $$= 1 - \big(P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)\big)$$
   $$= 1 - \big(0.6767 + 0.2707\big) = 0.0526.$$

   (d) Find the expected number of additional lifts that Happa-One will need to open on a given day. Call this number of lifts Y. Then

   $$E[Y] = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) + 2 \cdot P(Y = 2)$$
   $$= 0 \cdot 0.6767 + 1 \cdot 0.2707 + 2 \cdot 0.0526 = 0.3759.$$

2. A certain event happens, on average, 5 times per second.

   (a) Find the probability that this event happens 9 times in a given second.
   Let X be the number of times it happens in a second. Then

   $$P(X = 9) = \frac{5^9}{9!} e^{-9} = 0.0363.$$

   (b) Find the probability that it happens 18 times in a given two second interval.
   If it happens, on average, 5 times a second, then on average, it happens, on average,10 times every two seconds. So in this case, $\lambda = 10$. So if Y is the number of times it happens in a two second interval, Then

   $$P(Y = 18) = \frac{10^{18}}{18!} e^{-10} = 0.0071.$$

   (c) True or false: the event is just as likely to happen 18 times in two seconds as it is to happen 9 times in one second.
   False, because $P(Y = 18) = 0.0071$, which is much less than half of half of $P(X = 9) = 0.0363$.

3. Show that the formula **(∗)** for the Poisson distribution, on page 1, really does define a probability distribution. This means: show that all the probabilities add up to one, or in other words,

   $$\sum_{k=0}^{\infty} P(X = k) = 1.$$

   Hint: you may want to use the Taylor series for e^λ:

   $$e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

   By **(∗)**, we have

   $$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}.$$

   Since the factor $e^{-\lambda}$ doesn't depend on k, we can pull it outside of the sum. We get

   $$\sum_{k=0}^{\infty} P(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda},$$

   by the power series for e^λ. But $e^{-\lambda} e^{\lambda} = 1$, so we're done.

4. In discussing the Poisson distribution in class, we first came up with the formula

$$P(X = k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}, \qquad (**)$$

and then claimed that $(**)$ yields the formula $(*)$ on the first page of this review sheet. In class and in homework, we *proved* that $(**)$ yields $(*)$ for a few specific values of k. The purpose of this exercise is to show that this holds for *any* integer $k = 0, 1, 2, \ldots$.

(a) Use the definition $\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$, together with $(**)$, to show that

$$P(X = k) = \frac{\lambda^k}{k!} \lim_{N\to\infty} \frac{N!}{N^k\,(N-k)!} \left(1 - \frac{\lambda}{N}\right)^{N-k}.$$

By $(**)$, we have

$$\begin{aligned} P(X = k) &= \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k} \\ &= \lim_{N\to\infty} \frac{N!}{k!\,(N-k)!} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}. \end{aligned}$$

The factor λ^k in the numerator and the factor $k!$ in the denominator are independent of N, so we can bring them outside of the limit, to get

$$\begin{aligned} P(X = k) &= \frac{\lambda^k}{k!} \lim_{N\to\infty} \frac{N!}{(N-k)!} \left(\frac{1}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k} \\ &= \frac{\lambda^k}{k!} \lim_{N\to\infty} \frac{N!}{N^k(N-k)!} \left(1 - \frac{\lambda}{N}\right)^{N-k}. \end{aligned}$$

(b) Use your answer to the previous part of this problem to find a simple formula (with no limits in it) for $P(X = k)$. Hint: use the following limit formulas (which may be proved using calculus ideas, like l'Hôpital's rule):

$$\lim_{N\to\infty} \frac{N!}{N^k\,(N-k)!} = 1; \quad \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^N = e^{-\lambda}, \quad \lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^{-k} = 1.$$

Another hint: you know what your answer should be, by $(*)$.

By our previous answer and by the given limits, we have

$$\begin{aligned} P(X = k) &= \frac{\lambda^k}{k!} \lim_{N\to\infty} \frac{N!}{N^k(N-k)!} \left(1 - \frac{\lambda}{N}\right)^{N-k} \\ &= \frac{\lambda^k}{k!} \cdot 1 \cdot e^{-\lambda} \cdot 1 = \frac{\lambda^k}{k!} e^{-\lambda}, \end{aligned}$$

which agrees with $(*)$.