Some Poisson Random Variable Review Problems

Recall the following. Suppose a certain event happens, on average, λ times per interval of a given extent. Let X be the number of times that the event *actually* happens in such an interval. Then we say "X is Poisson of parameter λ," and we have the formula

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad (k = 0, 1, 2, 3, \ldots). \tag{$*$}$$

1. On a given day, the Happa-One ski resort in Hakuba, Japan opens additional lifts depending on how many buses from Tokyo arrive at the resort by 11 AM. If fewer than 3 buses arrive by 11 AM, they open no additional lifts. If 3 or 4 buses arrive by 11 AM, they open one additional lift. If 5 or more arrive by 11 AM, they open two additional lifts.

   Suppose, on average, 2 buses from Tokyo arrive at the resort by 11 AM.

   (a) Find the probability that, on a given day, Happa-One will open no additional lifts. Hint: this probability equals $P(X = 0) + P(X = 1) + P(X = 2)$, where X is the number of buses arriving by 11 from Tokyo.

   (b) Find the probability that, on a given day, Happa-One will open one additional lift.

   (c) Find the probability that, on a given day, Happa-One will open two additional lifts.

   (d) Find the expected number of additional lifts that Happa-One will need to open on a given day.

2. A certain event happens, on average, 5 times per second.

   (a) Find the probability that this event happens 9 times in a given second.

   (b) Find the probability that it happens 18 times in a given two second interval.

(c) True or false: the event is just as likely to happen 18 times in two seconds as it is to happen 9 times in one second.

3. Show that the formula **(∗)** for the Poisson distribution, on page 1, really does define a probability distribution. This means: show that all the probabilities add up to one, or in other words,

$$\sum_{k=0}^{\infty} P(X = k) = 1.$$

Hint: you may want to use the Taylor series for e^{λ}:

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}.$$

4. In discussing the Poisson distribution in class, we first came up with the formula

$$P(X = k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}, \qquad (**)$$

and then claimed that **(∗∗)** yields the formula **(∗)** on the first page of this review sheet. In class and in homework, we *proved* that **(∗∗)** yields **(∗)** for a few specific values of k. The purpose of this exercise is to show that this holds for *any* integer $k = 0, 1, 2, \ldots$.

(a) Use the definition $\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$, together with **(∗∗)**, to show that

$$P(X = k) = \frac{\lambda^k}{k!} \lim_{N\to\infty} \frac{N!}{N^k\,(N-k)!} \left(1 - \frac{\lambda}{N}\right)^{N-k}.$$

(b) Use your answer to the previous part of this problem to find a simple formula (with no limits in it) for $P(X = k)$. Hint: use the following limit formulas (which may be proved using calculus ideas, such as l'Hôpital's rule):

$$\lim_{N\to\infty} \frac{N!}{N^k\,(N-k)!} = 1; \quad \lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^N = e^{-\lambda}, \quad \lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{-k} = 1.$$

Another hint: you know what your answer should be, by $(*)$.