MATH 4510: Intro to Probability

November 6, 2024

In-class Midterm Exam #2

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:_________________ SOLUTIONS _________________

Please show all work.

Please write neatly. If it's unreadable, it's ungradeable.

Use the backs of pages if you need more space.

Express *all* probabilities as decimals to at least four decimal places. (You can leave out trailing zeroes; e.g. you can write 0.3 for 0.3000 and 0.825 for 0.8250.)

If you get stuck on a problem, move on, and then come back to it.

Take a deep breath. Good luck!

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	12 pts	
2	25 pts	
3	20 pts	
4	20 pts	
5	7 pts	
6	20 pts	
TOTAL	100 pts	

1. (12 points; 6 points each) A 90% free throw shooter takes two free throws. Let X be the number of free throws made out of the two shots.

(a) Find the probability mass function for X. You can write down the general formula for $P(X = k)$ if you want, but also please compute and write down $P(X = 0)$, $P(X = 1)$, and $P(X = 2)$ individually as decimals.

$$P(X = 0) = \binom{2}{0} \cdot 0.9^0 \cdot 0.1^2 = 0.01,$$

$$P(X = 1) = \binom{2}{1} \cdot 0.9^1 \cdot 0.1^1 = 0.18,$$

$$P(X = 0) = \binom{2}{2} \cdot 0.9^2 \cdot 0.1^0 = 0.81.$$

(b) You pay \$10 to enter a game where a 90% free throw shooter takes two free throws. If 0 free throws go in, you receive \$30, if 1 free throw goes in, you receive \$15, and if 2 free throws go in, you receive \$1. What is your expected payoff (amount received minus the \$10 you paid in)? Express your answer to the nearest cent. Should you play the game? Explain.

The expected payoff, in dollars, is

$$(30 - 10) \cdot 0.01 + (15 - 10) \cdot 0.18 + (1 - 10) \cdot 0.81 = -6.19.$$

You definitely shouldn't play, since your expected payoff is negative.

2. (25 points; 5 points each) A fair coin is flipped 4 times (assume that these trials are independent). Let X be the number of times a pair of consecutive flips lands the same (that is, both land heads or both land tails). (Remark: in an outcome like HHHT , we say $X = 2$, since in this case the first two flips are the same *and* the second two are. Similarly, the outcome TTTT would correspond to $X = 3$.)

In this problem, we compute $E[X]$ in one way. In the next problem, we compute it a different way.

Note that the sample space is

$$S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,$$
$$THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}.$$

(a) Find $P(X = 0)$, by simply counting the outcomes, above, that have 0 pair of consecutive flips landing the same, and dividing by the size of the sample space.

$$P(X = 0) = \frac{\left|\{HTHT, THTH\}\right|}{16} = \frac{2}{16} = 0.125.$$

(b) Similarly, find $P(X = 1)$.

$$P(X = 1) = \frac{\left|\{HHTH, HTHH, HTTH, THHT, THTT, TTHT\}\right|}{16} = \frac{6}{16} = 0.375.$$

(c) Similarly, find $P(X = 2)$.

$$P(X = 2) = \frac{\left|\{HHHT, HHTT, HTTT, THHH, TTHH, TTTH\}\right|}{16} = \frac{6}{16} = 0.375.$$

(d) Similarly, find $P(X = 3)$.

$$P(X = 3) = \frac{\left|\{HHHH, TTTT\}\right|}{16} = \frac{2}{16} = 0.125.$$

(e) Using the general formula for expected value, find $E[X]$.

$$E[X] = \sum_{k=1}^{3} k \cdot P(X = 3) = 0 \cdot 0.125 + 1 \cdot 0.375 + 2 \cdot 0.375 + 3 \cdot 0.125 = 1.5.$$

3. (20 points; 4 points each) As in the previous problem, a fair coin is flipped 4 times (assume that these trials are independent), and X is the number of times a pair of consecutive flips lands the same (that is, both land heads or both land tails). We compute $E[X]$ as follows.

(a) What's the probability that the first two flips land the same? Please explain.

$$P(\text{both the same}) = \frac{\left|\{HH, TT\}\right|}{4} = 0.5.$$

(b) Let X_1 equal 1 if the first two flips land the same, and 0 if not. What is $E[X_1]$? Explain. Hint: for a Bernoulli random variable, the expected value equals the probability that the event in question happens.

$$E[X_1] = p = 0.5.$$

(c) Similarly, let X_2 equal 1 if the second and third flips land the same, and 0 if not. What is $E[X_2]$? Explain.

$$E[X_2] = p = 0.5.$$

(d) Similarly, let X_3 equal 1 if the third and fourth flips land the same, and 0 if not. What is $E[X_3]$? Explain.

$$E[X_3] = p = 0.5.$$

(e) Find $E[X]$ by expressing X in terms of X_1, X_2, and X_3, and using the sum rule for expected values. Show your work.

$$E[X] = E[X_1] + E[X_2] + E[X_3] = 3 \cdot 0.5 = 1.5.$$

4. (20 points; 4 points each) For this problem recall that, if X is a Poisson random variable with parameter λ, then we say "X is $P(\lambda)$."

In October, leaves fall off a certain tree at an average rate of 3 leaves per hour. (Assume the leaves behave independently of each other.)

(a) Let X be the number of leaves that fall from this tree in a random one-hour period in October. Then X is $P(\underline{\ \ 3\ \ })$ (fill in the blank with a number).

(b) Find $P(X = 0)$.

$$P(X = 0) = \frac{3^0}{0!}e^{-3} = 0.0498.$$

(c) Find $P(X = 1)$.

$$P(X = 1) = \frac{3^1}{1!}e^{-3} = 0.1494.$$

(d) Find $P(X = 2)$.

$$P(X = 2) = \frac{2^0}{2!}e^{-3} = 0.2240.$$

(e) Find $P(X > 2)$.

$$P(X > 2) = 1 - \big(P(X = 0) + P(X = 1) + P(X = 2)\big) = 0.5768.$$

5. (7 points) For the same tree as in problem 4 above, now let Y be the number of leaves that fall from the tree in a *two hour* period.

Find $P(Y = 2)$. Use any method you would like, but please explain your reasoning.

Since X (from the previous problem) is $P(3)$, Y is $P(6)$, so

$$P(Y = 3) = \frac{6^3}{3!}e^{-6} = 0.0892.$$

Alternatively,

$$\begin{aligned} P(Y = 3) &= P(X = 0)P(X = 3) + P(X = 1)P(X = 2) + P(X = 2)P(X = 1) + P(X = 3)P(X = 0) \\ &= 2\big(P(X = 0)P(X = 3) + P(X = 1)P(X = 2)\big) \\ &= 2\big(0.0498 \cdot 0.2240 + 0.1494 \cdot 0.2240\big) = 0.0892. \end{aligned}$$

6. (20 points; 4 points each) For this problem recall that, if Z is a binomial random variable with n trials, with probability of success p for each trial, then we say "Z is $B(n, p)$."

As in problem 4 above, consider a tree losing leaves at an average rate of 3 leaves per hour. But this time, suppose there are only 30 leaves left on the tree. Let Z denote the number of leaves that will be lost over the course of the next hour. Then Z is a binomial random variable, since each leaf can either fall or not. (Assume the leaves behave independently of each other.)

(a) Think of each leaf as a trial. Then our random variable Z consists of $n =$ __30__ trials. (Fill in the blank with a number.)

(b) Since, on average, 3 leaves are expected to fall each hour, we have $E[Z] =$ __3__ (fill in the blank with a number). (You don't need to use a formula. Just use the fact that expected value means what you expect on average.)

(c) Think of a "success" in a given trial (leaf) as being "the leaf falls." What is $p = P(\text{success})$? Hint: on your fact sheet, you have a formula for the expected value of a binomial random variable. Use this formula, together with the answers to parts (a) and (b) of this problem, to find p.

$E[Z] = 3 = np = 30 \cdot p$, so $p = 0.1$.

(d) To summarize your results above, Z is $B($__30__, __0.1__$)$. (Fill in the two blanks.)

(e) Using binomial probabilities, find $P(Z = 2)$. Hint: your answer should be similar to (though perhaps not exactly equal to) one of your answers from problem 4 above.

$$P(Z = 2) = \binom{30}{2} \cdot 0.1^2 \cdot 0.9^{28} = 0.2277.$$