MATH 4510: Intro to Probability

October 2, 2024

In-class Midterm Exam #1 (SOLUTIONS)

1. (12 points; 6 points each) (The two parts of this question are more or less unrelated, except that they involve one of Dr. S's favorite numbers, 10.)

(a) Dr. S. has 10 pairs of sneakers, which he wants to divide into a group of 5 pairs (for weekdays), a group of 2 pairs (for the weekend), and a group of 3 pairs (for emergencies). In how many ways can he do this? Please express your answer as a single integer (that is, a whole number, like 73 or 1,802,479). (Pairs of sneakers cannot be broken up; each pair stays together as a pair.)

The number of ways is

$$\binom{10}{5,3,2} = \frac{10!}{5!\,3!\,2!} = 2{,}520.$$

(b) Two fair, 6-sided dice are rolled. Find the probability that the sum on the dice is 10. Please express your answer as a fraction (like 1/11), or as a decimal to four places (like 0.7733), or as a percent to two places (like 77.33%).

$$P(\text{sum is }10) = \frac{|\{46,55,64\}|}{36} = \frac{3}{36} = \frac{1}{12} = 0.0833 = 8.33\%.$$

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2. (36 points; 6 points each)

Please do not round off any answers in this problem!

Consider events E, F, and G, in a sample space S, such that

$$P(F) = 0.2, \qquad P(E \cup F) = 0.5, \qquad P(G) = 0.3, \qquad P(F|G) = 0.6.$$

Assume that E and F are *mutually exclusive*, and that E and G are *independent*.

(a) Find $P(E)$. (Again, E and F are mutually exclusive.)

$$P(E) = P(E \cup F) = P(F) = 0.5 - 0.2 = 0.3.$$

(b) Using the number you found for $P(E)$ in part (a) above, find $P(EG)$. (Again, E and G are independent.)

$$P(EG) = P(E)P(G) = P(F) = 0.3 \cdot 0.3 = 0.09.$$

(c) Find $P(FG)$. (Do *not* assume that F and G are mutually exclusive, or independent.)

$$P(FG) = P(G)P(F|G) = 0.3 \cdot 0.6 = 0.18.$$

(d) Find $P(EF)$. (Again, E and F are mutually exclusive.) E and F are mutually exclusive, so they can't both happen together, so $P(EF) = 0$.

(e) Find $P(EFG)$. (Again, E and F are mutually exclusive.)
E and F are mutually exclusive, so they can't both happen together, certainly E, F, and G can't all happen together, so $P(EFG) = 0$.

(f) Use your answers to the various parts above to find $P(E \cup F \cup G)$.

$$\begin{aligned} P(E \cup F \cup G) &= P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) \\ &= 0.3 + 0.2 + 0.3 - 0 - 0.09 - 0.18 + 0 = 0.53. \end{aligned}$$

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3. (20 points; 5 points each) **Please do not round off any answers in this problem!**

Consider a certain basketball player's first two shot attempts in a game.

Define the following events:

B: makes both shots; F: makes the first shot but not the second;
S: makes the second shot but not the first; N: makes neither shot.

Suppose it's known that

$$P(B) = 0.36 = 36\%, \qquad P(F) = 0.24 = 24\%, \qquad P(N) = 0.16 = 16\%.$$

(a) What is the $P(S)$, the probability that this player makes the second shot but not the first?
We must have $P(B) + P(F) + P(S) + P(N) = 1$, so

$$P(S) = 1 - P(B) - P(F) - P(N) = 1 - .36 - .24 - .16 = .24.$$

(b) What is the probability that the player makes exactly one of the first two shots?

$$P(\text{exactly one shot}) = P(F) + P(S) = .24 + .24 = .48.$$

(c) If the basketball player (same player from the previous page) hits *both* of their first two shots, they have an 80% chance of hitting their third. If they hit *exactly one* of their first two shots, they have an 70% chance of hitting their third. If they *miss* both of their first two shots, they have a 50% chance of hitting their third.

Find $P(T)$, the probability that the player hits their third shot. Hint: use the generalization of Bayes's formula (the very last formula on your fact sheet).

$$\begin{aligned} P(T) &= P(\text{hits both})P(T|\text{hits both}) \\ &\quad + P(\text{hits exactly one})P(T|\text{hits exactly one}) + P(\text{misses both})P(T|\text{misses both}) \\ &= .36 \cdot .8 + .48 \cdot .7 + .16 \cdot .5 = .704 = 70.4\%. \end{aligned}$$

(d) Find the probability that the player makes all of their first three shots. Hint: you're looking for $P(TB)$, where B and T are as above. Use one of the formulas for $P(EF)$ from Part II on page 2 of your formula sheet.

$$P(TB) = P(B)P(T|B) = .36 \cdot .8 = 0.288 = 28.8\%.$$

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4. (16 points; 4 points each) For this problem, recall that a standard deck of cards has 52 cards: there are 13 face values (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A), and each face value has four suits (hearts, diamonds, spades, and clubs).

Draw 5 cards at random from a standard deck. We compute the probability of getting exactly one ace, as follows.

(a) How many possible 5-card hands (without keeping track of order) are there? Express your answer in terms of factorials and/or binomial coefficients.

$$\binom{52}{5}$$

(b) How many ways are there of choosing one ace? Express your answer as an integer.

$$4$$

(c) How many ways are there (without keeping track of order) of choosing four cards out of all the non-aces? Express your answer in terms of factorials and/or binomial coefficients.

$$\binom{48}{4}$$

(d) Use the above information to compute $P(\text{exactly one ace})$. Express your answer as a decimal to four places.

$$P(\text{exactly one ace}) = \frac{4 \cdot \binom{48}{4}}{\binom{52}{5}} = 0.2995.$$

5. (16 points; 4 points each) For this problem, we draw 5 cards at random from a standard deck, just as in the previous problem. As in that problem, we compute the probability that exactly one of these cards is an ace, but in a different way. This time, think of a 5-card hand as a *list*; that is, order matters. Here's how it works (the first answer is done for you, to show you the idea):

(a) First of all, out of 52 cards total, what is the total number of ways of *arranging* 5 cards, one after the other, in a list? **ANSWER:** $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$.

(b) Now let's choose an ace: how many ways are there of doing this? Express your answer as an integer.

$$4$$

(c) Now let's choose 4 non-aces from the 48 non-aces total, and arrange these cards, one after the other, in a list. How many ways are there of doing this? Express your answer as a product of integers, and/or as a single integer.

$$48 \cdot 47 \cdot 46 \cdot 45$$

(d) We're not done yet! The ace can go into any one of the 5 places on our list: We need to decide which of the 5 places into which to put this ace. How many ways are there to do this? (Remember that you've already chosen the ace, so the issue now is simply where to put that chosen ace.) Express your answer as an integer.

$$5$$

(e) There's nothing left to choose! (Once we've chosen where to put the ace, we know where to put the four non-aces.) So use the above information to compute $P(\text{exactly one ace})$. (You should get the same answer as in part (d) of problem 4 above). Express your answer as a decimal to four places.

$$P(\text{exactly one ace}) = \frac{4 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 5}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = 0.2995.$$