Applications of SDM II: Confidence intervals.

Let X have mean μ and std. dev. σ;
let $n \geq 30$; let

$$\bar{X} = \{\text{means of size-}n\text{ samples from } X\}.$$

SDM says that, under mild conditions, the "test statistic"

$$Z = \frac{\bar{X} - \mu}{(s/\sqrt{n})}$$

is roughly $N(0,1)$.

So for example, in such a case,

$$P\left(-2.58 < \frac{\bar{X} - \mu}{(s/\sqrt{n})} < 2.58\right) = 99\%.$$

Now inside of the $P(\ \ )$, do some algebra to get μ by itself in the middle:

(a) Multiply everything through by $-s/\sqrt{n}$.

(b) Add $\bar{X}$ to each term.

The result is:

$$P\left(\bar{X} - 2.58\frac{s}{\sqrt{n}} < \mu < \bar{X} + 2.58\frac{s}{\sqrt{n}}\right) = 99\%.$$

Interpretation: 99% of the time, when a size-n random sample from X is chosen, and

$\bar{x}$ and s are computed, the interval

$$(\bar{x}-2.58\, s/\sqrt{n},\ \bar{x}+2.58\, s/\sqrt{n}) \qquad (CI_{99})$$

will contain the true population mean μ.

The interval (CI_{99}) is called a 99% confidence interval for μ.

<u>Example.</u> In studying the Etruscan Empire (~700–300 BC), anthropologists measure the breadth of a "random" sample of $n = 84$ male Etruscan skulls, and find that

$$\bar{x} = 143.77 \text{ mm}, \quad s = 5.97 \text{ mm}.$$

Construct a 99% confidence interval for mean male Etruscan skull breadth μ.

<u>Solution.</u>
The interval is

$$\left(143.77 - 2.58 \cdot \frac{5.97}{\sqrt{84}},\ 143.77 + 2.58 \cdot \frac{5.97}{\sqrt{84}}\right)$$

$$= (142.09,\ 145.45).$$

<u>Notes</u> on (CI_{99}) in general.

1) Note that the interval (CI_{99}) is centered about the sample mean $\bar{x}$.

2) A 95% or 98% confidence interval for μ would look like (CI_{99}), but

with 2.58 replaced by 1.96 or 2.33 respectively.

3) So: <u>more</u> <u>confidence</u> requires a <u>wider</u> interval.

<u>Example 2.</u>

Let μ be as in Example 1. Test the null hypothesis

$$H_0 : \mu = 132.44 \text{ mm}$$

against the alternative hypothesis

$$H_A : \mu \neq 132.44 \text{ mm}$$

at the 99% level.

[Note: 132.44 is the mean skull breadth of present-day Italian males. So this test helps answer whether Etruscans were native to Italy.]

<u>Solution:</u>

We compute the z-statistic

$$z = \frac{\bar{x} - \mu_0}{(s/\sqrt{n})} = \frac{143.77 - 132.44}{(5.97/\sqrt{84})}$$

$$= 17.39.$$

Since $|z| > 2.58$, we reject H_0, and accept H_A, at the 99% level.

Note: in general,

"reject $H_0: \mu = \mu_0$, and accept $H_A: \mu \neq \mu_0$, at the p % level"

is the same as

"μ_0 is <u>outside</u> the p % confidence interval for μ."