In this activity, we compute probabilities is several ways.

Please express all probabilities as decimals to four decimal places.

A certain washing machine loses, on average, 10% of all socks washed in a given load. Assume socks are lost independently of each other.

1. Let X be the number of socks lost in a load containing 40 socks. Then X is a binomial random variable.

   (a) Find $E[X]$ and $\mathrm{Var}[X]$.

   $$E[X] = np = 40 \cdot 0.1 = 4, \qquad \mathrm{Var}[X] = np(1-p) = 40 \cdot 0.1 \cdot 0.9 = 3.6.$$

   (b) Find $P(X = 2)$.

   $$P(X = 2) = \binom{40}{2} \cdot (0.1)^2 \cdot (0.9)^{38} = 0.1423.$$

2. Now suppose the 40 socks above comprise 30 blue socks and 10 red socks.

   Let B denote the number of blue socks lost, and R the number of red socks lost.

   (a) Find $E[B]$, $\mathrm{Var}[B]$, $E[R]$, and $\mathrm{Var}[R]$. Again, assume independence. The sums $E[B]+E[R]$ and $\mathrm{Var}[B]+\mathrm{Var}[R]$ should look familiar.

   $$E[B] = np = 30 \cdot 0.1 = 3, \qquad \mathrm{Var}[B] = np(1-p) = 30 \cdot 0.1 \cdot 0.9 = 2.7,$$
   $$E[R] = np = 10 \cdot 0.1 = 1, \qquad \mathrm{Var}[R] = np(1-p) = 10 \cdot 0.1 \cdot 0.9 = 0.9.$$

(b) Find the probability that $B = 2$ *and* $R = 0$.

$$P(B=2)\cdot P(R=0)$$
$$= \left(\binom{30}{2}\cdot(0.1)^2\cdot(0.9)^{28}\right)\cdot\left(\binom{10}{2}\cdot(0.1)^2\cdot(0.9)^{8}\right) = 0.0082.$$

(c) Find the probability that $B = 1$ *and* $R = 1$.

$$P(B=1)\cdot P(R=1)$$
$$= \left(\binom{30}{1}\cdot(0.1)^1\cdot(0.9)^{29}\right)\cdot\left(\binom{10}{1}\cdot(0.1)^1\cdot(0.9)^{9}\right) = 0.0547.$$

(d) Find the probability that $B = 0$ *and* $R = 2$.

$$P(B=0)\cdot P(R=2)$$
$$= \left(\binom{30}{0}\cdot(0.1)^0\cdot(0.9)^{30}\right)\cdot\left(\binom{10}{2}\cdot(0.1)^2\cdot(0.9)^{8}\right) = 0.0794.$$

3. Using your answers to parts (b,c,d) of problem 2 above, find the probability that exactly two socks out of the 40 are lost. You should get a familiar answer.

$$0.0082 + 0.0547 + 0.0794 = 0.1423.$$

4. We now re-interpret problem 1 in terms of Poisson random variables.

   (a) Let Y be the number of socks lost in a load containing 40 socks. Since the machine loses, on average, 10% of all socks washed, the average number of socks lost in a load containing 40 socks is ___4___. In other words, Y is Poisson with parameter ___4___.

   (b) Use your answer to part (a) of this problem to find $P(Y = 2)$.

$$P(Y = 2) = \frac{4^2}{2!}e^{-4} = 0.1465.$$

   (c) Use your answer to part (a) of this problem to find $E[Y]$ and $\text{Var}[Y]$.

$$E[Y] = \text{Var}[Y] = 4.$$

5. You may have noticed that, even though X and Y are counting the same thing, $P(X = 2) \neq P(Y = 2)$, although they're close. Do you have any thoughts as to why this may be?

   Y is really only approximately a Poisson random variable. You may recall that, in deducing the formula for the probability mass function of a Poisson random variable, we assumed that we could break the interval in question down *ad infinitum*. But our "interval" here is 40 socks, and 40 socks cannot be broken down into infinitesimal bits! (Or maybe they can, but we're not assuming that here, we're assuming that only a whole number of socks can be lost.)

   In other words, Y is not *really* a Poisson random variable. (But, as demonstrated above, it is similar to one.)