Poisson, continued.

1. Examples.

Last time we argued that, if an event happens, on average, λ times per interval of a given length, and X is the number of times it happens on a random interval of that length, then

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}. \qquad (A)$$

(We say "X is $P(\lambda)$.")

Example 1.

A CPU receives an average of 3 instructions per nanosecond. It will crash if it receives >15 instructions in a nanosecond.

What's the probability of a crash?

Solution.

Let X be the number of instructions received in a nanosecond. Then

$$\begin{aligned} P(\text{crash}) &= P(X>15) \\ &= 1 - P(X \le 15) \\ &= 1 - \sum_{k=0}^{15} \frac{3^k \cdot e^{-3}}{k!} \\ &= 1.2408 \times 10^{-7}. \end{aligned}$$

Example 2.

Last time, we claimed that (A) followed from the formula

$$P(X=k) = \lim_{N\to\infty} \binom{N}{k}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k}. \qquad (B)$$

Use (B) to verify (A) in the case $k=1$.
Use the facts that, for any real number x,

(1) $\lim_{N\to\infty} (1+x/N)^N = e^x$;

(2) $\lim_{N\to\infty} (1+x/N)^{-1} = 1$.

Solution

By (B) with $k=1$,

$$P(X=1) = \lim_{N\to\infty} \binom{N}{1}\left(\frac{\lambda}{N}\right)^1\left(1-\frac{\lambda}{N}\right)^{N-1}.$$

$$= \lim_{N\to\infty} N\cdot\frac{\lambda}{N}\cdot\left(1-\frac{\lambda}{N}\right)^{N-1}$$

$$= \lim_{N\to\infty} \lambda\cdot\left(1-\frac{\lambda}{N}\right)^{N-1}. \qquad (C)$$

But

$$\left(1-\frac{\lambda}{N}\right)^{N-1} = \left(1-\frac{\lambda}{N}\right)^N\left(1-\frac{\lambda}{N}\right)^{-1}$$

which, by (1) and (2) with $x=-\lambda$, becomes $e^{-\lambda}\cdot 1 = e^{-\lambda}$ as $N\to\infty$. So (C) gives

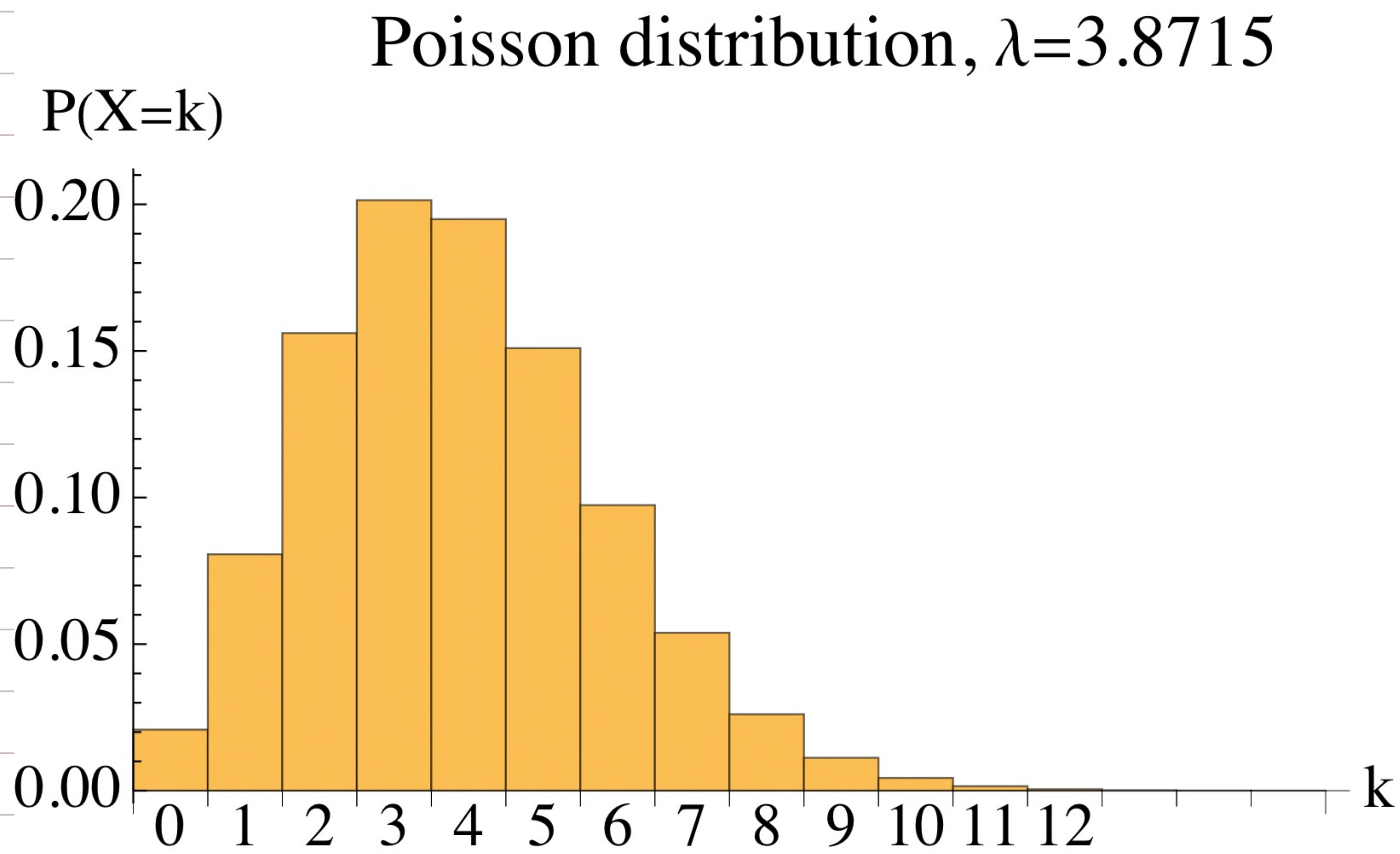
$$P(X=1) = \lambda e^{-\lambda},$$

which is the correct formula (A) for $k=1$.

B) Expected value and variance.

Last time, we considered $P(X=k)$ $(0 \le k \le 5)$ for a variable X that is $P(3.8715)$.

A histogram for $P(X=k)$ appears to be "centered" somewhere around $X=4$.



This reflects:

Theorem.

If X is $P(\lambda)$, then $E[X] = \lambda$.

Proof.

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k)$$

$$= 0 \cdot e^{-\lambda} + 1 \cdot \lambda e^{-\lambda} + 2 \cdot \frac{\lambda^2}{2!} e^{-\lambda} + 3 \cdot \frac{\lambda^3}{3!} e^{-\lambda} + 4 \cdot \frac{\lambda^4}{4!} e^{-\lambda} + \dots$$

note that $\frac{k}{k!} = \frac{1}{(k-1)!}$

$$= \lambda e^{-\lambda} + \frac{\lambda^2}{1!} e^{-\lambda} + \frac{\lambda^3}{2!} e^{-\lambda} + \frac{\lambda^4}{3!} e^{-\lambda} + \ldots$$

factor out $\lambda e^{-\lambda}$

$$= \lambda e^{-\lambda} \underbrace{\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right)}_{\text{this is the power series for } e^{\lambda}!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

Fact: it's also the case that
$\mathrm{Var}[X] = \lambda$.