The pmf for a binomial random variable.

Terminology: if X is a binomial rv with n trials and $P(\text{success}) = p$, we say X is $B(n,p)$.

Question: if X is $B(n,p)$, what is

$$P(X=k) \quad (0 \le k \le n)?$$

Answer: think of it this way.

(a) First, there are $\binom{n}{k}$ ways of distributing, or placing, the k successes among the n trials.

(b) For each of these ways, each of the k successes has probability p, while each of the $n-k$ failures has probability $1-p$. They're all independent, so their probabilities multiply.

(c) Putting it all together:

> If X is $B(n,p)$, then
>
> $$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (0 \le k \le n).$$

Example 1.

Flip 6 fair coins; let X = number of heads. Then

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}$$

$$= \frac{6!}{2!4!} \cdot \left(\frac{1}{2}\right)^6 = \frac{6 \cdot 5}{2 \cdot 1} \cdot \frac{1}{64} = \frac{15}{64} = .2344.$$

(see activity from 9/6).

Example 2. Repeat Example 1 with 6 unfair coins, for which P(heads) = .4.

Solution.

$$P(X=2) = \binom{6}{2}(.4)^2(.6)^4 = 0.3110.$$

Example 3.

A pea plant appears yellow if both of its color genes are yellow (call this pure yellow), or if one is yellow and one is green (call this hybrid). If both color genes are green (call this pure green), then the plant appears green.

Four pea plants are bred from two hybrid parent plants. Let X be the number of yellow (pure or hybrid) plants resulting.

Find:
(a) The pmf for X;
(b) $E[X]$; (c) $Var[X]$.

(Assume yellow and green genes are equally likely to be handed down.)

Solution. Define a success to be a yellow plant.

Write yg if a yellow gene is inherited from Parent 1 and a green gene from Parent 2, and so on.
The outcomes gg, yg, gy, yy are equally likely, so each has probability 1/4.
So

So: $p = P(\text{success}) = P(\{yg, gy, yy\}) = 3/4$.

(a) $P(X=0) = \binom{4}{0}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^4 = 0.0039,$

$P(X=1) = \binom{4}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^3 = 0.0469,$

$P(X=2) = \binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 = 0.2109,$

$P(X=3) = \binom{4}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^1 = 0.4219,$

$P(X=4) = \binom{4}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^0 = 0.3164.$

(b) $E[X] = np = 4 \cdot 3/4 = 3.$

(c) $Var[X] = np(1-p) = 4 \cdot \frac{3}{4} \cdot \frac{1}{4} = 0.75.$

Example 4.
An 80% free throw shooter makes 15 attempts in a game. Write $p(k)$ for P(exactly k of the 15 go in).

(a) Find $E[X]$.

(b) Without any computation, arrange in ascending order:

$$p(11),\ p(12),\ p(13).$$

Explain.

Solution.

(a) $E[X] = 15 \cdot .8 = 12.$

(b) Since $E[X] = 12$, making 12 shots seems especially likely, so $p(12)$ should be largest.

Also, 12 is about 92.31% of 13, while 11 is only about 91.67% of 12, so in a sense, 12 is closer to 13 than 11 is to 12. So we might expect that $p(13)$ is closer to $p(12)$ than $p(11)$ is.

Guess: $p(11) < p(13) < p(12)$.

(DIY: verify this.)