Bernoulli and binomial random variables.

A. Definition of Bernoulli rv.

Consider an experiment with only two possible outcomes, called a "success" and a "failure." Such an experiment is a Bernoulli experiment.

Examples:

1) Flip a coin: define "success" to be "coin lands heads."
2) Take a free throw: define "success" to be a made shot.
3) Crossbreed pea plants: assuming only green or yellow plants can result, define "success" to be a green plant resulting.

Now, given a Bernoulli experiment, define $p = P(\text{success})$. Also define an rv X by

$$X = \begin{cases} 1 & \text{if the experiment is a success;} \\ 0 & \text{if not.} \end{cases}$$

(So: X is the number of successes in a single trial of the experiment.)

X is called a Bernoulli rv.

Example 1.

Given a Bernoulli rv X with $P(\text{success}) = p$, find

(a) $E[X]$,

(b) $Var[X]$.

Solution.

(a) $E[X] = \sum_{\substack{\text{values} \\ x \text{ of } X}} x \cdot P(X=x)$

$$= 0 \cdot P(X=0) + 1 \cdot P(X=1)$$
$$= 0 \cdot (1-p) + 1 \cdot p = p.$$

(b) Write $\mu = E[X] = p$. Then

$$Var[X] = \sum_{\substack{\text{values } x \\ \text{of } X}} (x-\mu)^2 P(X=x)$$

$$= (0-p)^2 \cdot P(X=0) + (1-p)^2 \cdot P(X=1)$$
$$= p^2(1-p) + (1-p)^2 \cdot p$$
$$= p(1-p) \cdot (p + 1-p)$$
$$= p(1-p) \cdot 1$$
$$= p(1-p).$$

factor out $p(1-p)$

Conclusion:

for a Bernoulli rv with $P(\text{success}) = p$,

$E[X] = p$, $\quad var[X] = p(1-p)$.

Example 2.

An 80% free throw shooter takes one shot; let X = number of hits. Find $E[X]$ and $var[X]$.

Solution.

$E[X] = p = .8$, $\quad var[X] = p(1-p) = .8 \cdot .2 = .16$.

B. Definition of binomial rv.

Given n independent trials of a Bernoulli experiment, let X = number of successes. Then X is a <u>binomial</u> rv.

<u>Example 3</u>

Given a binomial rv X with n trials and $P(\text{success}) = p$ (for a single trial), find
(a) $E[X]$,
(b) $Var[X]$.

<u>Solution</u>.

(a) Let
$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ trial is a success} \\ 0 & \text{if not.} \end{cases}$$

Then $X = X_1 + X_2 + \dots + X_n$, so by the sum rule,

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$
$$= \underbrace{p + p + \dots + p}_{n \text{ times}}$$
$$= np.$$

(b) (A sum rule for variance.)
It's a <u>fact</u> (proof omitted) that, if $X_1, X_2, \dots, X_n$ are <u>independent</u> rv's, then

$$Var[X_1 + X_2 + \dots + X_n] = Var[X_1] + Var[X_2] + \dots + Var[X_n].$$

So, with all rv's as in (a) above,

$$\begin{aligned} Var[X] &= Var[X_1] + Var[X_2] + \dots + Var[X_n] \\ &= p(1-p) + p(1-p) + \dots + p(1-p) \\ &= np(1-p). \end{aligned}$$

Conclusion:
for a binomial rv with n trials and $P(\text{success}) = p$,

$$E[X] = np, \qquad Var[X] = np(1-p).$$

__Example 4.__
If an 80% free throw shooter takes 3 shots and X is the number made then, assuming independence,

$$E[X] = 3 \cdot .8 = 2.4,$$
$$Var[X] = 3 \cdot .8 \cdot .2 = .48.$$