Variance.

Definition 1: Given an rv X with expected value $\mu = E[X]$, we define the variance $\mathrm{Var}(X)$ of X by

$$\mathrm{Var}(X) = E[(X-\mu)^2] = \sum_{\substack{\text{values } x \\ \text{of } X}} (x-\mu)^2 P(X=x). \qquad (*)$$

Example 1.

Again, let Z be the number of made shots in 3 free throw attempts by an 80% shooter. We've seen that

$$P(Z=0) = 0.008, \quad P(Z=1) = 0.096, \quad P(Z=2) = 0.384,$$
$$P(Z=3) = 0.512, \quad \mu = E[Z] = 2.4.$$

So

$$\mathrm{Var}[Z] = \sum_{z=0}^{3} (z-\mu)^2 P(Z=z)$$
$$= (0-2.4)^2 \cdot 0.008 + (1-2.4)^2 \cdot 0.096$$
$$+ (2-2.4)^2 \cdot 0.512 + (3-2.4)^2 \cdot 0.512.$$
$$= 0.48.$$

Note: by $(*)$, $\mathrm{Var}[X]$ is a sort of average (weighted by probabilities) of $(x-\mu)^2$, which measures "how far x is from the expected value." So: $\mathrm{Var}[X]$ measures how spread out the values of X are.

Example 2.

Consider now a 50% free throw shooter taking 3 shots; let Y be the number made.

(a) Intuitively, which is larger, Var[Y] or Var[Z] (with Z as above)?

(b) Compute Var[Y].

<u>Solution</u>

(a) A 50% shooter's shots would, intuitively, be more spread out than an 80% shooter's shots (at 80%, there's more certainty as to where the shots will go). So we expect Var[Y] > Var[Z].

(b) First, we compute (DIY) that

$P(Y=0) = 1/8, \quad P(Y=1) = 3/8, \quad P(Y=2) = 3/8,$

$P(Y=3) = 1/8, \quad \mu = E[Y] = 1.5.$

So, by (*),

$$E[Y] = \sum_{y=0}^{3} (y-\mu)^2 P(Y=y)$$
$$= (0-1.5)^2 \cdot 1/8 + (1-1.5)^2 \cdot 3/8$$
$$+ (2-1.5)^2 \cdot 3/8 + (3-1.5)^2 \cdot 3/8$$
$$= 0.48.$$

(So yes, E[Y] > E[X].)

<u>Example 3.</u>

Random variables A and B both have possible values 1, 2, 3. Also, we have

$P(A=1) = .25, \quad P(A=2) = .5, \quad P(A=3) = .25,$

$P(B=1) = .4, \quad P(B=2) = .2, \quad P(B=3) = .4.$

(a) Intuitively, which is larger, $E[A]$ or $E[B]$?
(b) Compute $E[A]$ and $E[B]$.
(c) Intuitively, which is larger, $Var(A)$ or $Var(B)$?
(d) Compute $Var(A)$ and $Var(B)$.

Solution.
(a) Neither. They're both evenly distributed around the value 2.

(b) We have

$$E[A] = 1\cdot .25 + 2\cdot .5 + 3\cdot .25$$
$$= 2,$$

and

$$E[B] = 1\cdot .4 + 2\cdot .2 + 3\cdot .4$$
$$= 2.$$

(c) $Var(B)$, because its values are more spread out (less clustered around the middle value).

(d) We have

$$Var[A] = (1-2)^2\cdot .25 + (2-2)^2\cdot .5 + (3-2)^2\cdot .25$$
$$= .5,$$

and

$$Var[B] = (1-2)^2\cdot .4 + (2-2)^2\cdot .2 + (3-2)^2\cdot .4$$
$$= .8$$