Some properties of expected value.

0. Recall: if X is an rv, the probability mass function (pmf) for X is defined to be the function

$$P(X=x) \quad (x \in \{\text{possible values of } X\}).$$

Also, the expected value of X is defined by

$$E[X] = \sum_{\substack{\text{all values} \\ x \text{ of } X}} x \cdot P(X=x).$$

1. The sum rule. Fact:

for **any** rv's X and Y,

$$E[X+Y] = E[X] + E[Y].$$

More generally,

$$E[X_1+X_2+X_3+\ldots] = E[X_1]+E[X_2]+E[X_3]+\ldots$$

for **any** rv's $X_1, X_2, X_3, \ldots$

(Corollary 9.2, section 4.9)

Examples

A) An 80% free throw shooter makes 3 attempts; let X be the number that go in. Find $E[X]$.

Solution

Write $X = X_1+X_2+X_3$, where

$$X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ shot goes in,} \\ 0 & \text{if not} \end{cases}$$

$(1 \le i \le 3)$. Then clearly, for each i,

$$E[X_i] = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = 0.8.$$

So by the sum rule,

$$E[X] = E[X_1] + E[X_2] + E[X_3] = 3 \cdot 0.8 = 2.4.$$

[Note: we're <u>not</u> assuming independence, but we <u>are</u> assuming the shooter shoots the same percentage on all 3 shots.]

B) A fair 6-sided and a fair 10-sided die are rolled; let Z be the sum on the dice. Find $E[Z]$.

<u>Solution.</u>

Let X and Y denote the numbers on the 6- and 10-sided dice respectively. We've seen that $E[X] = 3.5$. Similarly we find that $E[Y] = 5.5$. So

$$E[Z] = E[X] + E[Y] = 3.5 + 5.5 = 9.$$

C) In a randomly shuffled standard deck of cards, how many adjacent pairs would we expect to have the same suit?

Solution.

Denote this number of pairs by X.

If we define rv's $X_1, X_2, X_3, \ldots, X_{51}$ by

$$X_i = \begin{cases} 1 & \text{if the } i^{\text{th}} \text{ and } (i+1)^{\text{st}} \text{ cards share a suit;} \\ 0 & \text{otherwise,} \end{cases}$$

then

$$X = \sum_{i=1}^{51} X_i .$$

Also, note that $E(X_i) = 12/51$ (whatever suit the i^{th} card is, 12 of the remaining 51 cards have the same suit).

So

$$E(X) = \sum_{i=1}^{51} E(X_i) = 51 \cdot 12/51 = 12.$$

2) Expected value of a function of X.

FACT:

> if X is an rv and g is a function, then
> $$E[g(X)] = \sum_{\substack{\text{all values} \\ x \text{ of } X}} g(x) \cdot P(X=x).$$

Proposition 4.1 (Section 4.4)

Example.

You pay \$6 to roll 2 fair, 6 sided dice. You receive $(X-7)^2$ dollars, where X is the sum on the dice. Should you play?

Solution. Your net winnings are

$$g(X) = (X-7)^2 - 6 \qquad \text{(in dollars).}$$

So by the above FACT,

$$E[g(X)] = \sum_{i=1}^{12} g(i)\, P(X=i) = \sum_{i=1}^{12} \left((i-7)^2 - 12\right) \cdot P(X=i).$$

Now we can compute that

$P(X=2) = P(X=12) = 1/36$, $P(X=3) = P(X=11) = 2/36$,
$P(X=4) = P(X=10) = 3/36$, $P(X=5) = P(X=9) = 4/36$,
$P(X=6) = P(X=8) = 5/36$, $P(X=7) = 6/36$.

So $E[g(X)] = \left((2-7)^2 - 6\right) \cdot 1/36 + \left((3-7)^2 - 6\right) \cdot 2/36$

$+ \dots + \left((12-7)^2 - 6\right) \cdot 1/36$

$= -1/6,$

so you shouldn't play.