Random variables and expected value.

Recall: a random variable (rv) is a function on a sample space.
For now, all rv's are discrete (i.e. take on only finitely or countably many values).

Given an rv X, we define the probability mass function (pmf) for X to be the function

$$P(X=x)$$

(as x ranges over possible values of X).

Example 1.
An 80% free throw shooter makes three attempts. Let Z be the number made.

We've seen that:

$P(Z=0) = 0.008$ $\quad$ $P(Z=2) = 0.384$
$P(Z=1) = 0.096$ $\quad$ $P(Z=3) = 0.512$

We also saw that

$$P(Z=z) = \binom{3}{z}(.8)^{z}(.2)^{3-z} \quad (0 \le z \le 3).$$

Definition: Given an rv X, we define the expected value $E[X]$ of X by

$$E[X] = \sum_{\substack{\text{all values} \\ x \text{ of } X}} x \cdot P(X=x)$$

The idea (think about it): $E[X]$ is "what we expect X to be on average."

Example 2 For Z as above,

$$E[Z] = \sum_{\substack{\text{all values} \\ z \text{ of } Z}} z \cdot P(Z = z)$$

$$= 0 \cdot 0.008 + 1 \cdot 0.096 + 2 \cdot 0.384 + 3 \cdot 0.512$$
$$= 2.4.$$

Example 3.

Roll a fair die; let N be the number showing. Find $E[N]$.

Solution

$$E[N] = \sum_{n=1}^{6} n \cdot P(N=n)$$
$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$
$$= \frac{1}{6}(1+2+3+4+5+6)$$
$$= 3.5.$$

Example 4.

Roll two fair dice; let T = the sum of the two numbers. Find

(a) the pmf for T;

(b) $E[T]$.

Solution

(a) The sample space has size 36. The possible values of T are $2, 3, \ldots, 12$. We computes for examples that

$$P(T=2) = \frac{|\{11\}|}{36} = 1/36,$$

$$P(T=5) = \frac{|\{14, 23, 32, 41\}|}{36} = 4/36,$$

$$P(T=10) = \frac{|\{46, 55, 64\}|}{36} = 3/36, \quad \text{etc.}$$

Using all these numbers, we find that

$$E[T] = \sum_{t=2}^{12} P(T=t) = \frac{1}{36} + \ldots + \frac{4}{36} + \ldots + 3/36 + \ldots$$

DIY: fill in missing numbers

$$= 7.$$

Example 5. Here's a card game:

You draw a card at random from a standard, 52-card deck. You win \$8 if you draw a face card or an ace; you lose \$3 if you draw a 2 through an 8; you lose \$6 if you draw a 9 or a 10.

Should you play?

Solution.

Let W be the rv of how much is won ($W<0$ for a loss).

Then, in dollars,

$$E(W) = 8 \cdot P(\text{win } \$8) - 3 \cdot P(\text{lose } \$3) - 6 \cdot P(\text{lose } \$6)$$

$$= 8 \cdot \frac{4}{13} - 3 \cdot \frac{7}{13} - 6 \cdot \frac{2}{13} = \frac{32-21-12}{13} = -1/13.$$

You'd expect to lose, so you shouldn't play.

Example 6.

A fair coin is flipped until it lands tails. Let X be the rv that records how many flips it took.

(a) Find the probability mass function for X.

(b) Find $E(X)$.

Solution.

To say that it takes i flips for the coin to land tails is to say that:

(1) The first $i-1$ flips land heads;

(2) The i^{th} lands tails.

Each outcome of a single flip has prob. $\frac{1}{2}$, so

$$P(X=i) = (\tfrac{1}{2})^i.$$

So

$$E(X) = \sum_{i=1}^{\infty} i\,P(X=i)$$

$$= \sum_{i=1}^{\infty} i \cdot (\tfrac{1}{2})^i = 2.$$

use Wolfram Alpha, Mathematica, Calculus, etc