Recall that a standard deck of cards has 13 hearts, and 39 cards that aren't hearts.

We draw three cards *without replacement* at random from a standard deck. That is: each card that is drawn is left out of the deck before drawing the next card. (This is the usual way cards are drawn or dealt.)

We are going to consider the probability that exactly one of the three cards is a heart, in two different ways.

Please write all answers as fractions or as decimals to at least four places.

1. In this method, we are, essentially, keeping track of the order in which the cards are dealt.

   (a) Find
   $$P(\text{first card is a heart and the other two are not}),$$
   using the multiplication rule, which says (in this context):
   $$\begin{aligned}&P(\text{first card is a heart and the other two are not})\\&= P(\text{first card is a heart}) \cdot P(\text{second card is } \textit{not} \text{ a heart given that the first is})\\&\cdot P(\text{third card is } \textit{not} \text{ a heart given that the first is and the second is not}).\end{aligned}$$
   You don't have to write things out in words. Just write in the numbers for each of the three individual probabilities in question, and then write in the result.

   $$\begin{aligned}&P(\text{the first card is a heart and the other two are not})\\&= P(\text{the first card is a heart}) \cdot P(\text{second card is not a heart given that the first is})\\&\cdot P(\text{third card is not a heart given that the first is and the second is not})\\&= \frac{13}{52} \cdot \frac{39}{51} \cdot \frac{38}{50} = 0.145294 \approx 14.53\%.\end{aligned}$$

(b) Find

$$P(\text{the second card is a heart and the other two are not}),$$

using a strategy much like the one you just used above.

$$P(\text{the second card is a heart and the other two are not}) = \frac{13}{52} \cdot \frac{39}{51} \cdot \frac{38}{50}$$
$$= 0.145294 \approx 14.53\%.$$

(c) Find $P(\text{the third card is a heart and the other two are not})$.

$$P(\text{the third card is a heart and the other two are not}) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{13}{50}$$
$$= 0.145294 \approx 14.53\%.$$

(d) Use your above results to find

$$P(\text{exactly one of the three cards is a heart}).$$

$$P(X = 1) = 0.145294 + 0.145294 + 0.145294 = 0.435882 \approx 43.59\%.$$

2. This time, we compute

$$P(\text{exactly one of the three cards is a heart})$$

without keeping track of order, by answering the following questions.

(a) How many ways are there of drawing 3 cards total out of 52 total (without keeping track of order)?

$$\binom{52}{3}$$

(b) How many ways are there of drawing one heart out of the 13 hearts in the deck?

$$\binom{13}{1}$$

(c) How many ways are there of drawing 2 non-hearts out of the 39 non-hearts in the deck (without keeping track of order)?

$$\binom{39}{2}$$

(d) Use the information from parts (a,b,c) of this problem to find

$$P(\text{exactly one of the three cards is a heart}).$$

Note: your answer should agree with your answer to 1(d) above.

$$\frac{\binom{13}{1}\binom{39}{2}}{\binom{52}{3}} = 0.435882 \approx 43.59\%.$$

Yup, it agrees with the above.

(e) **Thought experiment.** Suppose we drew our cards *with replacement* instead of without. That is, after each draw, the card is returned to the deck (and the deck is shuffled thoroughly) before the next card is drawn.

In this case, would

$$P(\text{exactly one of the three cards is a heart})$$

be higher or lower than it was above, where we drew without replacement? Answer without doing any computations! Just reason it out. Please explain your reasoning.

Drawing with replacement would make this probability lower. Why? Well there are more non-hearts than hearts in the deck, so on average, when you're putting a card back, you're putting back a non-heart, and therefore increasing the proportion of non-hearts in the deck, making the probability that the next card drawn is a heart a little bit lower.

In fact, one can compute that

$$P(\text{exactly one of the three cards is a heart, when drawing with replacement})$$
$$= 3\left(\frac{39}{52} \cdot \frac{13}{52} \cdot \frac{13}{52}\right) = 0.421875 \approx 42.19\%.$$